Physics 220- Fall 2011

Theory of Many Body Physics

Solution to Examination 2

Attempt any 4 questions

100 points, Time 1.5 hours   26 October, 2011

1. Electrons with spin half in 1 dimension live on a ring containing N sites. Find the relation between the number density of electrons $n = N_e/N$ and the Fermi momentum. What is the Fermi wave number at a density $n = 1/3$? [25]

   We can write the number of particles including a factor of two for spin as
$$n = \frac{N_e}{N} = \frac{2}{N}\sum_k \theta(k_F - |k|) \to 2\int_{-k_F}^{k_F} \frac{dk}{2\pi} = 2\frac{k_F}{\pi},$$
   hence at $n = 1/3$ we find $k_F = \pi/6$.

2. The above electrons have a hopping matrix element t_0 at nearest neighbor separation, and t_1 at second neighbor separation. Find t_1/t_0 such that the net dispersion has a flat bottom near $k \sim 0$ going as $\varepsilon_k = c_1 + c_2k^4$ (i.e. missing the quadratic piece). [25] *We can write the electronic dispersion as*
$$\varepsilon_k = -t_0\left[e^{ik} + e^{-ik}\right] - t_1\left[e^{i2k} + e^{-i2k}\right] = -2t_0\cos(k) - 2t_1\cos(2k),$$
   so expanding near $k \sim 0$ we get
$$\varepsilon_k = -2t_0 - 2t_1 + (t_0 + 4t_1)k^2 - \frac{1}{12}(t_0 + 16t_1)k^4 + O(k^6).$$
   Setting $t_1 = -1/4t_0$ we get the required result
$$\varepsilon_k = -(3/2)t_0 + \frac{1}{4}t_0k^4 + O(k^6).$$

3. Starting from the Fermionic version of the Jordan Wigner string
$$J[1,n] = e^{i\pi\sum_{j=1}^{n-1} C_j^\dagger C_j},$$
   show that
$$J[1,n]J[1,n] = 1.$$

[25]

Since the eigenvalues of $C_j^\dagger C_j$ are either 1 or 0, we see that the value of the operator

$$e^{i2\pi \sum_{j=1}^{n-1} C_j^\dagger C_j} \to 1.$$

4. With two Fermionic destruction operators C_j and $j = 1, 2$ with the canonical properties $\{C_i, C_j\} = 0$ and $\{C_i, C_j^\dagger\} = \delta_{ij}$, we construct two other Fermi operators:

$$\alpha = u\ C_1 + v\ C_2,$$
$$\beta = p\ C_1 + q\ C_2.$$

Write down the conditions on u, v, p, q in order that the new Fermions are canonical and mutually independent. [25]

Taking u, v as real we need to satisfy the conditions

$$\left\{\alpha, \alpha^\dagger\right\} = 1 = \left\{\beta, \beta^\dagger\right\};\ \ \left\{\alpha, \beta^\dagger\right\} = 0.$$

These lead to

$$u^2 + v^2 = 1 = p^2 + q^2,$$
$$u\ p + v\ q = 0.$$

The usual choice that achieves these is to write

$$u = \cos(\theta), v = \sin(\theta), p = -v, q = u.$$

5. For spin half electrons on a lattice, the spin density is given as $S_j^z = \frac{1}{2}(n_{i\uparrow} - n_{i\downarrow})$, and the number density as $\rho_j = (n_{i\uparrow} + n_{i\downarrow})$. Show that the Hubbard interaction term

$$U\ n_{i\uparrow}\ n_{i\downarrow},$$

is expressible as a suitable linear combination of $U(S_i^z)^2$, and the number density $U\rho_i$.[25]

Let us start with

$$(S_i^Z)^2 = \frac{1}{4}(n_{i\uparrow} - n_{i\downarrow})^2 = \frac{1}{4}(n_{i\uparrow}^2 + n_{i\downarrow}^2 - 2n_{i\uparrow}n_{i\downarrow}).$$

On using

$$n_{i\sigma}^2 = n_{i\sigma},$$

we get

$$(S_i^Z)^2 = \frac{1}{4}(\rho_i - 2n_{i\uparrow}n_{i\downarrow}).$$

Hence

$$U\ n_{i\uparrow}\ n_{i\downarrow} = -2U(S_i^Z)^2 + \frac{U}{2}\rho_i.$$