Physics 220- Fall 2011

Theory of Many Body Physics

Solution of Examination 3

100 points, Time 1.5 hours 21 November, 2011

1. (A) Starting with the (Matsubara type) Heisenberg equation of motion in the imaginary time formalism for an arbitrary operator Q:

$$\frac{d}{d\tau}Q(\tau) = [H, Q(\tau)],$$

calculate the time dependent Greens function

$$D(k,\tau) = \frac{1}{Z}\, Tr\, [e^{-\beta H}\ T_\tau \phi(k,\tau)\phi^+(k,0)]\ ,$$

in time domain for a variable

$$\phi(k,\tau) = \frac{1}{\sqrt{2}}\ (b(k,\tau) + b^+(-k,\tau)),$$

where $b(k)$ is a Bose destruction operator in a Free Bosonic theory with $H = \sum_k \varepsilon_k\ b^\dagger(k)b(k)$. (Here ε_k is an even function of k.)

[Hint: A judicious choice for Q helps here. Find the time dependence first and use the statistical mechanical information about averages of Bosonic operators.] [30]

Solution:

We calculate

$$\begin{aligned}
D(k,\tau) &= \frac{1}{Z}\, Tr\, [e^{-\beta H}\ T_\tau\ \phi(k,\tau)\phi^+(k,0)],\ (0)\\
&= \frac{1}{2Z}\, Tr\, [e^{-\beta H}\ T_\tau\ (b(k,\tau)b^+(k,0) + b^+(-k,\tau)b(-k,0))],\ (1)\\
&= \frac{1}{2Z}\, Tr\, [e^{-\beta H}\ T_\tau\ (b(k,\tau)b^+(k,0) + b(-k,-\tau)b^+(-k,0))],\ (2)\\
&= -\frac{1}{2}(G_B(k,\tau) + G_B(-k,-\tau)).\ (3)
\end{aligned}$$

In Step (1) we discarded terms such as $(b(k,\tau)b(-k,0)$ since their averages vanish in the ensemble average. In Step (2) we commuted the two Bose operators under the time ordering sign and used time translation invariance to shift the argument of $b(-k)$. The final answer is written in terms of the Bose Greens function $G_B(k,\tau)$.

From the EOM, we can find the time dependence of the bosonic operators

$$b(k,\tau) = e^{-\varepsilon_k \tau} b(k,0),$$
$$b^{+}(k,\tau) = e^{\varepsilon_k \tau} b^{\dagger}(k,0),$$

and hence

$$G_B(k,\tau) = -\theta(\tau)\langle b(k) b^{\dagger}(k)\rangle \; e^{-\varepsilon_k \tau} \; - \theta(-\tau)\langle b^{\dagger}(k) b(k)\rangle \; e^{-\varepsilon_k \tau}.$$

Expressing this in terms of the Bose function $n_B(k) = 1/(e^{\beta\varepsilon_k} - 1)$, we write

$$G_B(k,\tau) = -\left[\theta(\tau)(1 + n_B(k)) + \theta(-\tau) n_B(k)\right] \quad e^{-\varepsilon_k \tau}.$$

Similarly

$$G_B(-k,-\tau) = -\left[\theta(-\tau)(1 + n_B(k)) + \theta(\tau) n_B(k)\right] \quad e^{\varepsilon_k \tau}.$$

(B) Find the Fourier series representation of D. [10]

The Fourier transform can be found as follows:

$$\begin{aligned} D(k, i\Omega_n) &= \int_{-\beta}^{\beta} \frac{d\tau}{2} e^{i\Omega_n \tau} D(k,\tau), \\ &= -\frac{1}{2}(G_B(k, i\Omega_n) + G_B(-k, -i\Omega_n)) \end{aligned}$$

Using the periodicity (where $0 \leq \tau \leq \beta$)

$$G_B(k,\tau) = G_B(k, \tau - \beta),$$

we can write as an integral over positive τ only:

$$\begin{aligned} G_B(k, i\Omega_n) &= \int_0^{\beta} d\tau \; e^{i\Omega_n \tau} \; G_B(k,\tau), \\ &= \frac{1}{i\Omega_n - \varepsilon_k} \end{aligned}$$

Using this we write the final answer

$$\begin{aligned} D(k, i\Omega_n) &= \frac{1}{2}\left[\frac{1}{i\Omega_n + \varepsilon_k} - \frac{1}{i\Omega_n - \varepsilon_k}\right], \; (4) \qquad (1) \\ &= -\frac{\varepsilon_k}{\varepsilon_k^2 + \Omega_n^2}. \; (5) \end{aligned}$$

2. Using partial fractions or otherwise, calculate frequency sum over ω_p in the bubble susceptibility

$$\chi_0(\vec{Q}, i\Omega_\nu) = -(k_B T)\frac{1}{N_s} \sum_p G_0(p) G_0(p+Q),$$

where $G_0(p)$ is the free Fermionic Greens function and $Q = (\vec{Q}, i\Omega_\nu)$ is a Bosonic Matsubara frequency. (The answer will contain an unevaluated momentum sum.) [30]

We may write $G_0(p) = 1/(i\omega_p - \xi_p)$ and hence the partial fraction splitting

$$G_0(p)G_0(p+Q) = [G_0(p) - G_0(p+Q)] \; \frac{1}{i\Omega_Q + \xi_p - \xi_{p+Q}}.$$

We may insert a convergence factor $e^{i\omega_p 0^+}$ in the sum so that we can exploit the standard result

$$(k_B T) \sum_{\omega_p} e^{i\omega_p 0^+} G_0(p) \to f(p),$$

where $f(p)$ is the Fermi occupation probability. Hence we get the well known bubble susceptibility:

$$\chi_0(\vec{Q}, i\Omega_\nu) = -\frac{1}{N_s} \sum_{\vec{p}} \frac{f(p) - f(p+Q)}{i\Omega_Q + \xi_p - \xi_{p+Q}}.$$

3. A Bosonic Greens function $D(k, i\Omega_\nu)$ is given as

$$D(k, i\Omega_\nu) = -\frac{\varepsilon_k}{\varepsilon_k^2 + \Omega_\nu^2}.$$

Infer its spectral weight $\rho_D(k, \nu)$, where the relation between the two is given as:

$$D(i\Omega_k, k) = \int_{-\infty}^{\infty} d\nu \; \frac{\rho_D(k,\nu)}{i\Omega_k - \nu} \; d\nu.$$

[30]

[Hint Use standard complex variable theory of analytic continuation in the appropriate variable.]

Consulting Eqs (4,5) in Problem 1 (B), we immediately write

$$D(k, i\Omega_n) = \frac{1}{2}\left[\frac{1}{i\Omega_n + \varepsilon_k} - \frac{1}{i\Omega_n - \varepsilon_k}\right],$$

and hence the analytic continuation reads

$$D(k, z) = \frac{1}{2}\left[\frac{1}{z + \varepsilon_k} - \frac{1}{z - \varepsilon_k}\right],$$

Hence

$$\rho_D(k, \nu) = -\frac{1}{\pi}\ \Im m\ D(k, \nu + i0^+) = \frac{1}{2}\left[\delta(\nu + \varepsilon_k) - \delta(\nu - \varepsilon_k)\right].$$