Physics 210- Fall 2018

Classical and Statistical mechancis

Hamilton Jacobi Definitions Revisited
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§Definitions

$$\dot{q}p - H(q,p) = \dot{Q}P - K(Q,P) + \frac{d}{dt}S(q,P) \tag{1}$$

where $S(q,P) = F_2(q,P)$, and so by multiplying out with dt we get

$$dS(q,P) = (K-H)dt + pdq + QdP \tag{2}$$

So far this is general. We now ask for a transformation such that $K = 0$. Hence

$$p = \frac{\partial S(q,P)}{\partial q} \tag{3}$$

$$Q = \frac{\partial S(q,P)}{\partial P} \tag{4}$$

$$0 = H(q,p) + \frac{\partial S(q,P)}{\partial t} \tag{5}$$

and the equations of motion (henceforth EOM)

$$\dot{P} = 0 \tag{6}$$

$$\dot{Q} = 0. \tag{7}$$

We used $K = 0$ to obtain the above pair. We now assume that $H(q,p)$ is independent of time t. Let us now look at Eq(5) and plug in for p from Eq(3). This gives us a partial differential equation (PDE) for S

$$H(q, \frac{\partial S(q,P)}{\partial q}) + \frac{\partial S(q,P)}{\partial t} = 0 \tag{8}$$

Since the time dependence is only in the second term, we can separate this PDE into two pieces

$$S(q,P) = W(q,P) - E(P)t \tag{9}$$

$$H(q, \frac{\partial W(q,P)}{\partial q}) = E(P) \tag{10}$$

where Eq(10) is obtained by plugging in Eq(9) into Eq(8). Here $E(P)$ is as yet undetermined, it has dimensions of energy. Similarly $W(q, P)$ is undetermined as yet.

§Example of Harmonic oscillator

We can make some sense of the above equations Eq(8,9,10) by choosing

$$H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2}. \tag{11}$$

This implies from Eq(10)

$$\frac{kq^2}{2} + \frac{1}{2m}\left(\frac{\partial W(q,P)}{\partial q}\right)^2 = E(P), \tag{12}$$

and hence we can solve for W as

$$W(q,P) = \pm \int dq\, \sqrt{2m(E - q^2/2)} + C \tag{13}$$

where we must keep in mind that $E = E(P)$. We will solve this further below, but let us first redefine the variables a bit.

§Action variable and angle variable

Let us get rid of P in favor of the action variable J. If we consider a periodic motion with $p = p(E,q) = \sqrt{(2m)(E - V(q))}$, we can define the action variable

$$J(E) \equiv \oint p(E,q)dq \tag{14}$$

we can invert this and write

$$E = E(J). \tag{15}$$

An explicit and simple example is the Harmonic oscillator Eq(11), where

$$E(J) = \frac{\omega_0}{2\pi} J, \tag{16}$$

with $\omega_0 = \sqrt{k/m}$. We also saw the example of the anharmonic oscillator $H = p^2/2 + q^4/4$, where

$$E(J) = cJ^{4/3}.$$

Getting back to Eq(9), changing $P \to J$ and $Q \to \beta$ we rewrite it as

$$S(q,J) = W(q,J) - E(J)t \tag{17}$$

$$H(q, \frac{\partial W(q,J)}{\partial q}) = E(J) \tag{18}$$

and Eq(4) as

$$\beta = \frac{\partial S(q, J)}{\partial J}, \tag{19}$$

and Eq(6,7) become

$$\dot{J} = 0, \tag{20}$$
$$\dot{\beta} = 0. \tag{21}$$

Taking the J derivative of Eq(17), and by plugging in Eq(19) and moving terms around, we write

$$\frac{\partial W(q, J)}{\partial J} = \frac{\partial E(J)}{\partial J} + \beta \tag{22}$$

where β is a constant in time. we now define the angle variable $\theta(q, J)$ and frequency constant $\omega(J)$ using

$$\theta(q, J) \equiv (2\pi)\frac{\partial W(q, J)}{\partial J} \tag{23}$$
$$\omega(J) \equiv (2\pi)\frac{\partial E(J)}{\partial J} \tag{24}$$

Hence Eq(22) becomes

$$\theta(q, J) = \omega(J)t + (2\pi)\beta. \tag{25}$$

The final thing to check is the proof of angular change in a closed orbit $\Delta\theta$.

$$\begin{aligned}
\Delta\theta &= \oint dq \, \frac{\partial \theta}{\partial q} && (26)\\
&= (2\pi) \oint dq \, \frac{\partial^2 W(q, J)}{\partial q \partial J} && (27)\\
&= (2\pi) \oint dq \, \frac{\partial^2 W(q, J)}{\partial J \partial q} && (28)\\
&= (2\pi)\frac{\partial}{\partial J} \oint dq \, p \, dq && (29)\\
&= (2\pi)\frac{\partial}{\partial J} J = (2\pi). \ \textbf{QED} && (30)
\end{aligned}$$

We have used Eq(23) to get Eq(27), and then used Eq(3) with $S \to W$ in going from Eq(28) to Eq(29).