ADVANCES IN MATHEMATICS 16, 197–220 (1975)

Three Integrable Hamiltonian Systems Connected with Isospectral Deformations*

J. MOSER

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

DEDICATED TO STAN ULAM

1. INTRODUCTION

(a) *Background.* In the early stages of classical mechanics it was the ultimate goal to integrate the differential equations of motions explicitly or by quadrature. This led to the discovery of various "integrable" systems, such as Euler's two fixed center problems, Jacobi's integration of the geodesics on a three-axial ellipsoid, S. Kovalevski's motion of the top under gravity for special ratios of the principal moments of inertia, to name a few nontrivial examples. These efforts and their climax with the work of Jacobi who applied skillfully the method of separation of variables to partial differential equations, the Hamilton–Jacobi equations associated with the mechanical system, to establish their integrable character.

However, this development took a sharp turn when Poincaré showed that most Hamiltonian systems are not integrable and gave arguments indicating the nonintegrability of the three-body problem. In the same negative direction lies Brun's discovery that the three-body problem has no algebraic integral except for the well-known classical ones and algebraic functions of these. These results express, in other words, that integrability of Hamiltonian systems is not a generic property; it is destroyed under small perturbations of the Hamiltonian.

Therefore it seems an anachronismus to discuss these exceptional

* This work was partially supported by the National Science Foundation, Grant No. NSF-GP-42298X.

integrable systems nowadays. However, in recent years various phenomena were discovered which are clearly intimately related to integrable Hamiltonian systems yet they have very different origin. One is related to the discovery by Kruskal and others [6] of so-called solitons for the Korteweg–de Vries equation. These are wave solutions of a nonlinear partial differential equation having a strong stability behavior. Originally these phenomena were brought to light by numerical experiments and later on related to the existence of infinitely many conservation laws that restrict the evolution of the solutions severely. If one interprets the partial differential equation, in this case the Korteweg–de Vries equation, as a Hamiltonian system in an infinite-dimensional function space, with a certain symplectic structure, and the conservation laws as integrals of this system, one can view this as an example of an integrable system of infinitely many degrees of freedom. This was made precise in the work of Zakharov and Faddeev [15].

In an entirely unrelated development Calogero [2, 3] found that the quantum theoretical problem of n mass points on the line interacting under the influence of a potential proportional to the inverse square of the distance can be solved explicitly, and he conjectured that the corresponding classical problem might be integrable. This was established by Marchioro for the "three-body problem" by explicit calculation. Moreover, Calogero used his formula to study the scattering problem associated with the n-particle system in the quantum theoretical framework and found that the scattering is essentially trivial, in the sense that the particles behave asymptotically like elastically reflected mass points.

(b) *Results.* It is our goal to show a close algebraic connection between these so different problems. However, instead of studying the infinite dimensional problems related to the partial differential equation in the one and the quantum theoretical framework in the other case, we will restrict ourselves to finite-dimensional systems. The Korteweg–de Vries equation can be discretized so as to retain the desired integrability, as was shown by Toda [13] and his collaborators. Another discretization leads to the differential equations

$$du_k/dt = \tfrac{1}{2}(e^{u_{k+1}} - e^{u_{k-1}}), \qquad (k = 1, 2, ..., n-1) \tag{1.1}$$

(where we set formally $e^{u_0} = 0$, $e^{u_n} = 0$) suggested by M. Kac and P. v. Moerbeke [8, 9]. Although this system does not have the appearance of a Hamiltonian system, it can be embedded into one, as was shown in [12]. The remarkable fact is that there are $[n/2] = \nu$ polynomials P_μ of u_k, e^{u_k} which are integrals of the motion, i.e.,

$$dP_\mu/dt = 0 \qquad (\mu = 1, 2, ..., \nu)$$

if one inserts a solution of the above differential equations. Moreover, all solutions can be expressed in the form

$$e^{u_k} = R_k(\eta)$$

where R_k are rational functions of

$$\eta = (\eta_1, ..., \eta_\nu) \qquad \text{and} \qquad \eta_1 = e^{\alpha_1 t}, ..., \eta_\nu = e^{\alpha_\nu t}.$$

These rational functions can, of course, not be explicitly described, but this representation suffices to give a complete description of the scattering problem related to this problem (see Section 7).

Instead of Calogero's quantum theoretical problem we look at the corresponding classical one, described by the equations

$$d^2x_k/dt^2 = -(\partial U/\partial x_k), \qquad (k = 1, 2, ..., n)$$

where

$$U = \sum_{k<l} (x_k - x_l)^{-2}, \qquad k, l = 1, 2, ..., n, \tag{1.2}$$

the coordinates x_k of the mass points being distinct real numbers. This system is clearly a Hamiltonian system with

$$\mathscr{H} = \tfrac{1}{2} \sum_{k=1}^{n} y_k^2 + U$$

where y_k are the momenta. We will show that this system is an integrable Hamiltonian system, by which we mean that this system possesses n independent integrals $I_k = I_k(x, y)$, globally defined in the phase space and in involution. In this case these functions are, in fact, polynomials in y_k and $(x_k - x_l)^{-1}$. Using this result it is quite easy to verify Marchioro's conjecture: The particles have an asymptotic velocity $\dot{x}_k(\pm\infty)$ satisfying

$$\dot{x}_k(+\infty) = \dot{x}_{n+1-k}(-\infty).$$

Thus after a fairly complicated interaction the particles emerge as free particles with velocities exchanged, that is, the first particle has for $t \to +\infty$ the velocity of the last for $t \to -\infty$, etc.

As a third example we discuss the equation on the circle

$$d^2x_k/dt^2 = -(\partial U/\partial x_k)$$

with

$$U = \tfrac{1}{2} \sum_{k\neq l(\mathrm{mod}\, n)} \sin^{-2}(x_k - x_l), \tag{1.3}$$

Here the x_k are considered mod π as distinct points on a circle. These equations are the classical mechanics analog to those of Sutherland [14]. Also this system will be shown to be an integrable Hamiltonian system with n integrals I_k which are polynomials in y_k and $\cot(x_k - x_l)$.

In contrast to the previous examples the last problem has a compact energy surface. On account of this fact the surfaces $I_k = \text{const}$ $(k = 1, 2,..., n)$ are compact and hence, as is well known, tori on which the solutions are quasiperiodic. However, the function theoretical character of these solutions has not yet been satisfactorily described.

(c) *Lax's method.* The common link between these problems is that they can be related to deformations of matrices leaving the eigenvalues fixed, that is, to isospectral deformations. For example, with (1.2) we associate a Hermitean matrix L having y_k as diagonal elements and $i(x_k - x_l)^{-1}$ as elements in the (k, l)-position if $k \neq l$. Then (1.2) gives rise to a differential equation for $L = L(t)$ whose solutions have fixed eigenvalues, i.e., the eigenvalues, and hence their symmetric functions I_k are integrals of the motion. The idea of finding integrals of the motion as eigenvalues of an associated linear operator L was developed by Lax [10] for the Korteweg–de Vries equation, where L is given by the classical Sturm–Liouville operator

$$-(d^2/dx^2) + q$$

and the potential q is to be deformed in such a way that the spectrum is unchanged. This question is intimately related to the inverse problem of determining the spectrum from the potential. Instead of developing these ideas in generality we will illustrate them in the three simple examples mentioned above.

In Section 2 we illustrate this method for Eq. (1.1), although this is in no way new. Indeed Flaschka [4, 5] observed first that this method can be applied to the Toda lattice and this example is only a slight variation on this theme. In Section 3 we put Eq. (1.2) into the same framework and draw the conclusion for the associated scattering problem in

Section 4. The n-particle system (1.3) on the circle will be studied in Section 5. Finally in Sections 6 and 7 we discuss the inverse spectrum problem and the scattering problem associated with a special Jacobi matrix. The latter leads to an interesting motion in which particles separate in pairs, each pair having a different asymptotic velocity, while the two particles of one pair have the same asymptotic velocity. The scattering phases can also be determined by relating the differential equations to those for the Toda lattice for finitely many particles.

(d) *General remarks.* These problems have connections with a multitude of topics besides that of dynamical systems. The fact that they are related to isospectral deformation points to the connection with spectral and scattering theory. The function theoretical nature of the solution and the rational character of the integrals relates to functions of complex variables. But also Lie algebras play into the subject; in fact the equations are very similar in nature to those studied by Arnold [1]. Arnold generalized the Euler equation for the rotation of a rigid body to dynamical systems in arbitrary Lie algebra.

Many of these connections are still obscure, and we hope that the study of these simple finite dimensional examples will lead to further investigations clarifying the many questions left open.

I want to express my thanks to H. Flaschka and G. Galavotti for many stimulating discussions in the beginning of this work. I am particularly indebted to Galavotti who pointed out Calogero's work and insisted that the classical analog should be integrable.

2. Isospectral Deformations

We begin with an idea that was introduced by P. D. Lax in a different but closely related connection. Consider a class of matrices, say all Jacobi matrices of the form

$$L = \begin{pmatrix} 0 & a_1 & 0 & & & \\ a_1 & 0 & a_2 & & 0 & \\ \cdot & \vdots & \vdots & \ddots & & \\ & & \vdots & \vdots & \ddots & \\ & 0 & & \cdot & \vdots & a_{n-1} \\ & & & & a_{n-1} & 0 \end{pmatrix} \tag{2.1}$$

with positive entries $a_1, a_2, \ldots, a_{n-1}$. Their eigenvalues are real and simple. We ask for all matrices in this class having the same spectrum.

One may expect that there are not enough parameters available, but since

$$K^{-1}LK = -L \qquad \text{for} \quad K = \operatorname{diag}(1, -1, +1, \cdots),$$

one has for the characteristic polynomial

$$\Delta_n(\lambda) = \det(\lambda I - L)$$

the relation

$$\Delta_n(\lambda) = (-1)^n \Delta_n(-\lambda). \tag{2.2}$$

Therefore, with λ also $-\lambda$ is an eigenvalue and $\lambda = 0$ is an eigenvalue precisely if n is odd. Thus fixing the eigenvalues amounts to $[n/2]$ conditions and the dimensionality of the isospectral matrices of the form (2.1) is $n - [n/2]$.

To get some isospectral deformations, Lax [10] considered differential equations of the form

$$\frac{d}{dt} L = BL - LB \tag{2.3}$$

where $L = L(t)$, t being the deformation parameter. The matrix B has to be chosen appropriately, so that the commutator $[B, L]$ has zeros except in the two off-diagonals, and those should agree. In this example one finds as one possible choice the skew symmetric matrix

$$B = \begin{pmatrix} 0 & 0 & a_1a_2 & 0 & & & \\ 0 & 0 & 0 & a_2a_3 & & & \\ -a_1a_2 & 0 & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & 0 & a_{n-2}a_{n-1} \\ & & & & 0 & 0 & 0 \\ & & & & -a_{n-2}a_{n-1} & 0 & 0 \end{pmatrix}$$

for which the differential equation (2.3) takes the form

$$\dot{a}_k = a_k(a_{k+1}^2 - a_{k-1}^2), \qquad k = 1, 2, \ldots, n-1 \tag{2.4}$$

where we set formally $a_0 = 0 = a_n$.

It is clear that (2.3) gives rise to isospectral deformations: If we solve the differential equation

$$\frac{d}{dt} U = BU, \qquad U(0) = I$$

then (2.3) assures that

$$\frac{d}{dt}(U^{-1}LU) = 0,$$

hence

$$U^{-1}LU = L(0).$$

Thus the eigenvalues of L remain constant under this deformation. Also the coefficients I_k of the characteristic polynomial

$$\Delta_n(\lambda) = \lambda^n + I_1\lambda^{n-1} + \cdots + I_n$$

are integrals of the motion, which are polynomials in $a_1^2, a_2^2, \ldots, a_{n-1}^2$. By (2.2) only $\nu = [n/2]$ of these are not zero, but the remaining I_2, $I_4, \ldots, I_{2\nu}$ are actually independent polynomials.

With

$$a_k = \tfrac{1}{2}e^{\frac{1}{2}u_k}$$

the equations (2.4) take the form

$$\dot{u}_k = \tfrac{1}{2}(e^{u_{k+1}} - e^{u_{k-1}}) \qquad (k = 1, 2, \ldots, n-1) \tag{2.5}$$

where we formally set $u_0 = -\infty$, $u_n = -\infty$. These are the equations which Kac and v. Moerbeke considered in their discretization of the Korteweg–de Vries equation [8].[1] The above derivation is, of course, not new; it is quite analogous to that of Flaschka [4]. But we will use the above representation (2.3) of the differential equation (2.4) to describe its solutions as rational functions of exponentials (Section 6) and to investigate the scattering problem related to (2.5) (Section 7).

Incidentally, the above equations (2.3) do not represent the only deformations of L preserving the spectrum. On the contrary all B giving rise to such deformations form an $(n - [n/2])$-dimensional space [12].

3. The n-Particle System on the Line with the Inverse Square Potential

We consider n particles on the line with coordinates $x_1, x_2, \ldots, x_n$ and define

$$U(x) = \sum_{k<l} (x_k - x_l)^{-2}, \qquad k, l = 1, 2, \ldots, n \tag{3.1}$$

[1] As I learned from H. Flaschka, this system (2.5) and its relation to the Toda lattice was already mentioned by M. Hénon in a letter of August 28, 1973.

as their potential so that the equations of motion are given by

$$d^2x_k/dt^2 = -(\partial U/\partial x_k) = 2 \sum_{j\neq k} (x_k - x_j)^{-3} \qquad (k = 1, 2, \ldots, n). \tag{3.2}$$

It is remarkable that this system possesses n integrals of the motion which are polynomials in $\dot{x}_k$ and $(x_k - x_l)^{-2}$. This fact can again be derived by considering isospectral deformations of another class of matrices.

The quantum-mechanical analog of (3.2) has been studied by Calogero and Marchioro in a number of papers [2, 3, 11] and Calogero succeeded in determining explicit expressions for the spectrum for this problem. He conjectured from his work that the classical problem, being the limit of the quantum-theoretical one, should be integrable. For $n = 3$ this was already verified by Marchioro [11] but his approach does not lend itself to generalization. In order to introduce the class of matrices adapted to this problem we set

$$z_{kl} = \begin{cases} (x_k - x_l)^{-1} & \text{for} \quad k \neq l \\ 0 & \text{for} \quad k = l \end{cases}$$

and form the matrices

$$\begin{aligned} Z_\alpha &= (z_{kl}^\alpha) && \text{for} \quad \alpha = 1, 2, \\ Y &= \operatorname{diag}\{y_1, \ldots, y_n\}, \\ D_\alpha &= \operatorname{diag}\left\{\sum_{j=1}^n z_{kj}^\alpha\right\} && \text{for} \quad \alpha = 2, 3. \end{aligned} \tag{3.3}$$

Then we define

$$L = Y + iZ_1; \qquad B = iD_2 - iZ_2, \tag{3.4}$$

so that $L = L^*$ is Hermitean and B skew Hermitean.

The deformation equations

$$dL/dt = BL - LB \tag{3.5}$$

for this class of matrices can be transformed into the equation of motion (3.2)! This implies by the argument of the previous section that the coefficients I_k of the characteristic polynomial

$$\det(\lambda I - L) = \lambda^n + I_1\lambda^{n-1} + \cdots + I_n$$

are integrals of the differential equations. Moreover, they are rational functions of the coordinates and in involution.

To relate Eqs. (3.2) and (3.5) to each other observe that (3.5) depends only on the $n - 1$ differences $x_{k+1} - x_k$ ($k = 1, 2,..., n - 1$), while (3.2) involves all n coordinates x_k. Therefore we rewrite (3.2) in terms of the z_{kl} and $y_k = -\dot{x}_k$

$$\begin{aligned} \dot{y}_k &= -\ddot{x}_k = -2 \sum_{j=1}^{n} z_{kj}^3 \\ \dot{z}_{kl} &= z_{kl}^2(y_k - y_l). \end{aligned} \tag{3.6}$$

Of course this system is highly redundant, since only the $n - 1$ variables $z_{k,k+1}$ are independent, the other being determined by the relations

$$z_{kl}^{-1} = z_{kr}^{-1} + z_{rl}^{-1} \text{ if } k, l, r \text{ distinct, and } z_{kl} + z_{lk} = 0. \tag{3.7}$$

But one verifies immediately that these relations are consistent with (3.6): If they hold for $t = 0$ then for all t.

Now we identify (3.6) with the deformation equations (3.5). For this purpose we have to compute

$$[B, L] = i[Y, Z_2] - [D_2, Z_1] + [Z_2, Z_1] \tag{3.8}$$

where we used (3.4). The element of $[Z_2, Z_1]$ in the (k, l) position is given by

$$\sum_r (z_{kr}^2 z_{rl} - z_{rl}^2 z_{kr}),$$

hence the corresponding term in $[Z_2, Z_1] - [D_2, Z_1]$ is

$$\sum_r (z_{kr}^2 z_{rl} - z_{rl}^2 z_{kr}) - \sum_r (z_{kr}^2 - z_{rl}^2) z_{kl}.$$

To simplify this expression we use the identities (3.7) as follows: The summands of the sum above can be factored

$$Q_{kl,r} = z_{kr}^2 z_{rl} - z_{rl}^2 z_{kr} - (z_{kr}^2 - z_{rl}^2) z_{kl} = (z_{kr} - z_{rl}) P_{kl,r}$$

with

$$P_{kl,r} = (z_{kr} z_{rl} - (z_{kr} + z_{rl}) z_{kl}).$$

If all k, l, r are distinct, this takes the form

$$P_{kl,r} = z_{kr} z_{rl} z_{kl} \{z_{kl}^{-1} - z_{rl}^{-1} - z_{kr}^{-1}\} = 0$$

on account of (3.7). For $k \neq l$ one gets obviously

$$P_{kl,r} = -z_{kl}^2 \quad \text{if} \quad r = k \quad \text{or} \quad r = l.$$

Thus

$$\sum_{r=1}^{n} Q_{kl,r} = 0 \qquad \text{for} \quad k \neq l$$

which shows that $[Z_2, Z_1] - [D_2, Z_1]$ is a diagonal matrix. If one computes the diagonal elements one finds

$$[Z_2, Z_1] - [D_2, Z_1] = -2D_3,$$

with the notation of (3.3). Thus with (3.8) the equations (3.5) take the form

$$dL/dt = i[Y, Z_2] - 2D_3,$$

and, in components,

$$\dot{y}_k = -2 \sum z_{kj}^3,$$
$$\dot{z}_{kl} = (y_k - y_l)\, z_{kl}^2,$$

in agreement with (3.6).

This establishes the existence of the integrals, as well as their rational character. In Section 4, in which we study the scattering problem for this system, we will find without further calculation that these integrals are in involution.[2]

4. Asymptotic Behavior, Marchioro's Conjecture

The n-particle system of the preceding section has a very simple behavior. Since the particles exert a repelling force on each other they fly apart as $t \to \pm\infty$ and ultimately behave like force particles. From this it is clear that the limits $\lim_{t\to\infty} \dot{x}_k(\pm t) = \dot{x}_k(\pm\infty)$ exist. As a matter of fact, these limit velocities or their symmetric functions can be assigned as integrals to the orbits to which they belong. Thus the *existence* of integrals is no surprise for a system like (3.2). However, the existence of *rational* integrals is remarkable, and it implies that

$$\dot{x}_k(+\infty) = \dot{x}_{n+1-k}(-\infty), \qquad k = 1, 2, \ldots, n, \tag{4.1}$$

[2] Extending this method, M. Adler, a student at New York University, found n rational integrals for $U = \sum_{k<l} \{\alpha(x_k - x_l)^{-2} + \beta(x_k - x_l)^2\}$.

so that the particles simply exchange their velocity. Moreover, the above velocities are distinct and agree with the negative of the eigenvalues of the matrix (3.4) belonging to the orbit considered. This way we will prove the fact that matrices of the form (3.4) always have *simple* eigenvalues. One may ask for the phase shifts δ_k defined by

$$x_k(t) - x_{n-k+1}(-t) - 2\dot{x}_k(\infty)t \to \delta_k$$

for $t \to +\infty$. It is easily verified that $\delta_1 = \delta_2 = 0$ for $n = 2$, and one may conjecture that $\delta_k = 0$ for any $n > 2$, but this we have not been able to establish.[3]

The relations (4.1) have been established by Marchioro [11] for the case $n = 3$ and were conjectured by him for arbitrary n. For the quantum-mechanical problem they were established by Calogero [2].

To prove the above assertion we observe that we may label the particles according to the order

$$x_1 < x_2 < \cdots < x_n .$$

Indeed, since the Hamiltonian of (3.2) is given by

$$\mathscr{H} = \tfrac{1}{2} \sum_{k=1}^{n} y_k^2 + \sum_{k<l} (x_k - x_l)^{-2}, \tag{4.2}$$

the minimal existence of the particles is bounded away from zero for any solution. Moreover, the velocities $-y_k = \dot{x}_k$ are bounded for all t for every orbit.

Our next goal is to show that

$$\lim_{t\to\infty} y_k(\pm t) = y_k(\pm\infty) \tag{4.3}$$

exists and that

$$y_1(+\infty) > y_2(+\infty) > \cdots > y_n(+\infty). \tag{4.4}$$

From

$$\tfrac{1}{2}(\ddot{x}_n - \ddot{x}_1) = \sum_{j<n} (x_n - x_j)^{-3} + \sum_{j>1} (x_j - x_1)^{-3} > 0 \tag{4.4$'$}$$

and the boundedness of $\dot{x}_k$ we conclude, by integration that

$$\int_{-\infty}^{+\infty} (x_k - x_l)^{-3}\, dt < \infty \qquad \begin{array}{l} \text{for} \quad k > l = 1 \text{ and} \\ \text{for} \quad l < k = n. \end{array} \tag{4.5}$$

[3] *Note added in proof.* Meanwhile we have been able to verify that indeed $\delta_k = 0$ for all $n \geqslant q$.

Considering the other differential equations one concludes with a simple induction argument (which we forego) that (4.5) holds for all pairs $k > l$. This, in turn implies from (3.2) that the limits $\lim_{t\to\infty} \dot{x}_k(\pm t)$ exist, proving (4.3). Because of the ordering of the particles we have obviously

$$\begin{aligned} \dot{x}_1(+\infty) \leqslant \dot{x}_2(+\infty) \leqslant \cdots \leqslant \dot{x}_n(+\infty) \\ \dot{x}_1(-\infty) \geqslant \dot{x}_2(-\infty) \geqslant \cdots \geqslant \dot{x}_n(-\infty). \end{aligned} \tag{4.6}$$

To prove (4.4) we proceed as follows: Consider first $\phi(t) = x_n - x_1 > 0$ which, by (4.4′), satisfies

$$\tfrac{1}{2}\ddot{\phi} \geqslant 2(x_n - x_1)^{-3} > 0. \tag{4.7}$$

Thus $\dot{\phi}$ is monotone increasing and $\dot{\phi}(+\infty) \geqslant 0$, by (4.6). Were $\dot{\phi}(+\infty) = 0$ then $\dot{\phi}(t) < 0$ and thus ϕ bounded. But then the right-hand side of (4.7) would be bounded away from zero, hence ϕ unbounded. This contradiction shows that

$$\dot{x}_1(+\infty) < \dot{x}_n(+\infty).$$

Thus, in the first row of (4.6) we do not have equality in all places, i.e., there exists an s with

$$\dot{x}_s(+\infty) < \dot{x}_{s+1}(+\infty). \tag{4.8}$$

From this we will show now $\dot{x}_1(+\infty) < \dot{x}_s(+\infty)$ and $\dot{x}_{s+1}(+\infty) < \dot{x}_n(+\infty)$ which implies readily that all velocity are different. It suffices to show $x_1(+\infty) < x_s(+\infty)$, the other case being symmetric to it.

From (4.8) we conclude that $x_j - x_s = 0(t^{-1})$ for $j > s$ and therefore

$$\begin{aligned} \frac{1}{2}\frac{d^2}{dt^2}(x_s - x_1) &= \sum_{j<s} (x_s - x_j)^{-3} - 0(t^{-3}) + \sum_{j>1} (x_j - x_1)^{-3} \\ &\geqslant 2(x_s - x_1)^{-3} - 0(t^{-3}). \end{aligned}$$

Thus $\psi = x_s - x_1 + At^{-1}$ with some positive constant A satisfies

$$\ddot{\psi} \geqslant 4(x_s - x_1)^{-3} \qquad \text{for} \quad t > t_0$$

and is bounded from below. Thus $\dot{\psi}$ is increasing and $\dot{\psi}(+\infty) \geqslant 0$. As before we conclude that the assumption $\dot{\psi}(\infty) = 0$ leads to a contradiction. Since $\dot{\psi}(t) < \dot{\psi}(\infty) = 0$ $(t > t_0)$ implies ψ to be bounded for

$t > t_0$ hence $\ddot{\psi}$ would be bounded away from zero, and so ψ unbounded. Thus $\dot{\psi}(\infty) > 0$ as we wanted to show.

Since $y_k = -\dot{x}_k$ we have established (4.4). This implies obviously

$$(x_k - x_l)^{-1} = 0(t^{-1}) \qquad \text{for} \quad t \to +\infty, \quad k \neq l$$

so that we can see that the matrix $L(t)$ has a limit $L(\infty)$ which is a diagonal matrix. Since the eigenvalues λ_k of $L(t)$ are independent of t we have

$$y_k(+\infty) = \lambda_k$$

if we make the convention to order these like

$$\lambda_n < \lambda_{n-1} < \cdots < \lambda_1 .$$

For $t \to -\infty$ the matrix L also approaches a diagonal matrix with the same eigenvalues in the diagonal, but, because of (4.6) in reversed order. Thus

$$\dot{x}_k(+\infty) = -y_k(+\infty) = -\lambda_k; \quad \dot{x}_{n+1-k}(-\infty) = -y_{n+1-k}(-\infty) = -\lambda_k$$

and (4.1) is proven.

Finally, we observe that the integrals $I_k = I_k(x, y)$ ($k = 1, 2, \ldots, n$) are in involution. For $x_k - x_{k-1} \to \infty$ these integrals I_k converge with their derivatives to $\sigma_k(y)$, the symmetric functions of y. Thus the Poisson bracket

$$G_{kl} = \sum_{r=1}^{n} \frac{\partial(I_k , I_l)}{\partial(x_r , y_r)} = \{I_k , I_l\}$$

converges to $\{\sigma_k , \sigma_l\} = 0$. Thus, along any solution of our system $G_{kl} \to 0$ as $t \to \infty$. On the other hand, as is well known, G_{kl} are integrals themselves, hence $G_{kl} = 0$ for all x, y.

5. The Periodic Case—Sutherland's Equation

If one wants to study the problems of the previous two sections on the circle it is natural to use the identity

$$\sum_{k=-\infty}^{+\infty} (x - k\pi)^{-2} = \sin^{-2} x$$

as motivation to introduce the potential

$$U(x) = \tfrac{1}{2} \sum_{k \neq l(n)} \alpha^2 \sin^{-2}(\alpha(x_k - x_l)) \qquad (\alpha > 0) \tag{5.1}$$

where the summation is taken over all distinct pairs $k, l \pmod n$. The coordinates x_k of the particles may be defined for all integers k such that

$$x_k = x_l \,(\mathrm{mod}(\pi/\alpha)) \qquad \text{if and only if} \quad k = l \,(\mathrm{mod}\, n),$$

so that is suffices to consider x_k for $k = 1, 2, \ldots, n$. The differential equations take the form

$$d^2x_k/dt^2 = -(\partial U/\partial x_k) = 2\alpha^3 \sum_{j \neq k(n)} \cot \alpha(x_k - x_j) \sin^{-2}(\alpha(x_k - x_j)) \tag{5.2}$$

which is the classical analog of Sutherland's equation [14]. With $y_k = -\dot{x}_k$ the Hamiltonian is

$$\mathscr{H} = \frac{1}{2} \sum_{k (\mathrm{mod}\, n)} y_k{}^2 + \frac{\alpha^2}{2} \sum_{k \neq l(n)} \sin^{-2}(\alpha(x_k - x_l))$$

showing that, on an energy surface $\mathscr{H} = \text{const}$, the minimal distance of the particles remains bounded away from zero and the velocities $| y_k |$ bounded away from ∞. Thus the energy surface is compact and most solutions of (5.2) turn out to be quasi-periodic. This will be a consequence of well known facts [1] about integrable Hamiltonian systems if we show that (5.2) has n independent integrals which are in involution.

The construction of these integrals follows the pattern of Section 3. We set

$$\begin{aligned} z_{kl} &= \alpha \cot \alpha(x_k - x_l) \qquad && \text{if} \quad k \neq l(n), \\ z_{kl} &= 0 && \text{if} \quad k = l(n) \end{aligned}$$

and rewrite the system (5.2) in the form

$$\begin{aligned} \dot{y}_k &= U_{x_k} = -2 \sum_{j \neq k(n)} z_{kj}(\alpha^2 + z_{kj}^2) \\ \dot{z}_{kl} &= (\alpha^2 + z_{kl}^2)(y_k - y_l) \qquad \text{for} \quad k \neq l(n). \end{aligned} \tag{5.3}$$

Here the last line follows from the differential equation for $\cot x$.

To put these differential equations in the form (3.5) we introduce the n by n matrices

$$Z_1 = (z_{kl}); \qquad Z_2 = (z_{kl}^2 + \alpha^2)$$

where $k, l = 1, 2, \ldots, n$. With

$$D_2 = \operatorname{diag}\left\{ \sum_{j (\bmod n)} (z_{kj}^2 + \alpha^2) \right\}; \qquad D_3 = \operatorname{diag}\left\{ \sum_{j (\bmod n)} z_{kj}(z_{kj}^2 + \alpha^2) \right\}$$

$$Y = \operatorname{diag}\{y_k\}$$

we set

$$L = Y + iZ_1; \qquad B = iD_2 - iZ_2 . \tag{5.4}$$

Then it is a straightforward, though surprising, calculation that (5.3) can be written in the form

$$dL/dt = BL - LB. \tag{5.5}$$

In fact, for $\alpha \to 0$ the formal identities go over into those of Section 3, except for the boundary conditions.

Thus it follows that the coefficients $I_1, I_2, \ldots, I_n$

$$\det(\lambda I - L) = \lambda^n + I_1 \lambda^{n-1} + \cdots + I_n$$

are independent integrals of the motion. We will not verify here that they are in involution,[4] but observe that they are rational functions of y_k and $e^{i\alpha(x_k - x_l)}$.

To verify (5.5) one has to use the addition theorem for $\cot x$ which gives, for k, l, r distinct modulo n:

$$z_{kl} = \frac{z_{kr} z_{rl} - \alpha^2}{z_{kr} + z_{rl}}$$

hence, for $k \neq l \,(\bmod n)$

$$P_{kl,r} = z_{kr} z_{rl} - \alpha^2 - (z_{kr} + z_{rl})\, z_{kl} = \begin{cases} 0 & \text{if } \; r \neq k, l(n) \\ -z_{kl}^2 - \alpha^2 & \text{if } \; r = k, l(n). \end{cases}$$

This implies for

$$Q_{kl,r} = (z_{kr} - z_{rl})\, P_{kl,r} = -z_{kr}(z_{rl}^2 + \alpha^2) + z_{rl}(z_{kr}^2 + \alpha^2) - (z_{kr}^2 - z_{rl}^2)\, z_{kl}$$

that

$$\sum_r Q_{kl,r} \begin{cases} = 0 & \text{if } \; k \neq l \,(\bmod n) \\ = -2 \sum_r (z_{kr}^2 + \alpha^2) z_{kr} & \text{if } \; k = l \,(\bmod n) \end{cases}$$

[4] This could be done by replacing α by $i\alpha$ and using the same argument as in the previous section.

so that the matrix with the elements $\sum_r Q_{kl,r}$ agrees with the diagonal matrix $-2D_3$.

Now we compute the commutator

$$[Z_2 - D_2, Z_1] = \left(\sum_r Q_{kl,r}\right) = -2D_3$$

and, thus, from (5.4)

$$[B, L] = i[Y, Z_2] - [D_2, Z_1] + [Z_2, Z_1] = i[Y, Z_2] - 2D_3 .$$

From this identity one reads off that (5.5) agrees with the equation (5.3). This makes the statement about the I_k being integrals of the motion again obvious.

6. Rational Character of the Solution of (2.4)

We return to the equations (2.4) or (2.5) and investigate their solutions using the fact that these differential equations describe isospectral deformation of Jacobi matrices. We begin with introducing a set of variables r_k on the manifold of Jacobi matrices (2.1) for which the spectrum is fixed. This is the analog of the inverse spectrum problem.

Let

$$R(\lambda) = (\lambda I - L)^{-1}$$

and e_1 be the vector with components $(1, 0, ..., 0)$. We introduce the rational function

$$f(\lambda) = (R(\lambda)\, e_1 , e_1)$$

which has simple poles at $\lambda = \lambda_k$ with a positive residue which we denote by r_k^2, so that

$$f(\lambda) = \sum_{k=1}^{n} \frac{r_k^2}{\lambda - \lambda_k} .$$

On account of the symmetry property $K^{-1}LK = -L$ derived in Section 2, $f(\lambda)$ is an odd function of λ. Thus, if we order the (always distinct) eigenvalues by

$$\lambda_n > \lambda_{n-1} > \cdots > \lambda_1 \tag{6.1}$$

we conclude that

$$\lambda_k = -\lambda_{n-k+1}; \qquad r_k = r_{n-k+1}$$

and $f(\lambda)$ can be represented by

$$f(\lambda) = \sum_{k=1}^{\nu} \frac{2\lambda r_k^2}{\lambda^2 - \lambda_k^2} + \kappa_n \frac{r_{\nu+1}^2}{\lambda}$$

where $\kappa_n = 1$ for n odd, $\kappa_n = 0$ if n is even and $\nu = [n/2]$.

Since $f(\lambda) \sim \lambda^{-1}$ for $|\lambda| \to \infty$ we have

$$\sum_{k=1}^{n} r_k^2 = 1$$

and we prefer to free ourselves from the latter restriction by using the r_k as projective coordinates. Therefore we set

$$f(\lambda) = \frac{\sum_{k=1}^{n} r_k^2/(\lambda - \lambda_k)}{\sum_{k=1}^{n} r_k^2} = \frac{\sum_{k=1}^{\nu} 2\lambda r_k^2/(\lambda^2 - \lambda_k^2) + \kappa_n(r_{\nu+1}^2/\lambda)}{\sum_{k=1}^{\nu} 2r_k^2 + \kappa_n r_{\nu+1}^2}. \tag{6.2}$$

The n variables $r_1, r_2, \ldots, r_\nu, \kappa_n r_{\nu+1}, \lambda_1, \lambda_2, \ldots, \lambda_\nu$ can be used to describe the Jacobi matrix (2.1) uniquely up to scaling of the r_k. In fact, the squares a_k^2 ($k = 1, 2, \ldots, n-1$) of the elements in (2.1) can be expressed rationally in terms of those r_j, λ_j, $1 \leqslant j \leqslant \nu$ and $r_{\nu+1}$ if n is odd. The reason for this fact lies in the representation of $f(\lambda)$ as a continued function

$$f(\lambda) = \cfrac{1}{\lambda - \cfrac{a_1^2}{\lambda - \cfrac{a_2^2}{\ddots \, \cfrac{}{\lambda - \cfrac{a_{n-1}^2}{\lambda}}}}} \tag{6.3}$$

which goes back to Stieltjes (used also in [12]). Since the computation of the continued fraction from the partial fraction expression is a rational process one finds that

$$a_k^2 = R_k(r, \lambda) \tag{6.4}$$

where the R_k are rational functions, homogeneous of degree zero in the r_j and homogeneous of degree two in λ.

Moreover, (6.4) can be viewed as mapping which takes the domain

$$D = \{(\lambda, r), \lambda_1 > \lambda_2 > \cdots > \lambda_\nu \geqslant 0; r_j > 0 \ (j = 1, 2, ..., n - \nu)\}$$

into the domain onto

$$\tilde{D} = \{a_j > 0, j = 1, 2, ..., n - 1\}$$

in such a way that the pre-image of each point is precisely one ray $(\rho r, \lambda)$ with a scalar $\rho > 0$.

We will show that in these homogeneous coordinates the differential equations take the simple form

$$\dot{\lambda}_k = 0; \qquad \dot{r}_k = -\lambda_k^2 r_k , \tag{6.5}$$

so that, via (6.4) the a_k^2 appear as rational functions of exponentials $e^{-\lambda_1^2 t}, ..., e^{-\lambda_\nu^2 t}$.

To prove this assertion we introduce the eigenvectors $\phi(\lambda_j)$ of L which we normalize by

$$\phi_1(\lambda_j) = (e_1 , \phi(\lambda_j)) > 0; \qquad |\phi(\lambda_j)| = 1. \tag{6.6}$$

If L is a solution of (2.3) these eigenvectors become functions of t which evolve according to

$$\phi(\lambda_j , t) = U(t)\, \phi(\lambda_j , 0)$$

where $U(t)$ is the unitary matrix of Section 2. Thus the eigenvectors satisfy the differential equation

$$\frac{d\phi(\lambda_j , t)}{dt} = -B\phi(\lambda_j , t).$$

We compute the resulting differential equation for the first component $\phi_1 = (e_1 , \phi)$

$$\dot{\phi}_1 = -a_1 a_2 \phi_3$$

and use the equations resulting from $(L - \lambda)\phi = 0$

$$-\lambda\phi_1 + a_1\phi_2 = 0$$

$$a_1\phi_1 - \lambda\phi_2 + a_2\phi_3 = 0$$

to express ϕ_3 in terms of ϕ_1. One finds readily

$$a_1 a_2 \phi_3 = (\lambda^2 - a_1^2)\phi_1$$

so that the differential equations for ϕ_1 become

$$\dot{\phi}_1 = -(\lambda^2 - a_1^2)\phi_1 . \tag{6.7}$$

Finally, to show that the $\phi_1(\lambda_k)$ are proportional to the r_k we write the resolvent $R(\lambda)$ in terms of the eigenvectors obtaining

$$f(\lambda) = (R(\lambda)\, e_1 , e_1) = \sum_k \frac{(\phi(\lambda_k), e_1)^2}{\lambda - \lambda_k}$$

so that

$$\phi_1(\lambda_k) = \frac{r_k}{(\sum_{j=1}^n r_j^2)^{1/2}} .$$

Thus the differential equations (6.5) give

$$\dot{\phi}_1(\lambda_k) = -(\lambda_k^2 - \sum \lambda_j^2 r_j^2)\, \phi_1(\lambda_k).$$

It is easy to verify that

$$a_1^2 = \sum_j \lambda_j^2 r_j^2 \left(\sum_j r_j^2\right)^{-1}$$

and the second equations of (6.5) have been verified. The first equations of (6.5) are clear from the derivation.

Thus the solutions a_k^2 of (2.4) are rational functions of exponential functions. We describe the solution for $n = 4$. Computing the continued fraction of $f(\lambda)$ explicitly one finds

$$a_1^2 = \frac{\lambda_1^2 r_1^2 + \lambda_2^2 r_2^2}{r_1^2 + r_2^2}, \qquad a_2^2 = \frac{(\lambda_2^2 - \lambda_1^2)^2 r_1^2 r_2^2}{(\lambda_1^2 r_1^2 + \lambda_2^2 r_2^2)(r_1^2 + r_2^2)},$$

$$a_3^2 = \frac{\lambda_1^2 \lambda_2^2 (r_1^2 + r_2^2)}{\lambda_1^2 r_1^2 + \lambda_2^2 r_2^2} .$$

Inserting $r_j = r_j(0)\, e^{-\lambda_j^2 t}$ we obtain the explicit solutions of (2.4).

7. The Scattering Problem Associated with the Equation of Kac and v. Moerbeke

In order to study the asymptotic behavior of the solution of (2.5) we consider

$$u_k = x_k - x_{k+1}\,, \qquad k = 1, 2, \ldots, n-1 \tag{7.1}$$

as the difference between the positions x_k of n particles on the line. If the x_k satisfy the differential equations

$$\dot{x}_k = -\tfrac{1}{2}(e^{u_k} + e^{u_{k-1}}), \qquad k = 1, 2, \ldots, n \tag{7.2}$$

where we formally set $e^{u_0} = 0 = e^{u_n}$, or $x_0 = -\infty$, $x_{n+1} = +\infty$ then clearly (2.5) follows. Conversely the x_k are determined only up to translation and for any solution $x_k(t)$ of (7.2) also $x_k(t) + c$ is a solution giving rise to the same solution of (2.5), provided c is a constant. For simplicity we will assume that $n = 2\nu$ is even.

We ask for the asymptotic behavior of the solution of (7.2) for $t \to \pm\infty$ and the relation between the scattering data. We will show that any solution of (7.2) behaves linearly for large t:

$$x_k(\pm t) \sim \pm\alpha_k^{\pm} t + \beta_k^{\pm} \qquad \text{as} \quad t \to +\infty$$

where

$$\alpha_{2j}^{+} = \alpha_{2j-1}^{+} = \alpha_{n-2j+2}^{-} = \alpha_{n-2j+1}^{-}\,, \qquad j = 1, 2, \ldots, \nu, \tag{7.3}$$

i.e., the particles travel asymptotically in pairs, while the different pairs have negative and different velocities, in fact, it turns out

$$\alpha_{2j}^{+} = -2\lambda_j^2, \qquad j = 1, 2, \ldots, \nu \tag{7.3$'$}$$

where the $\lambda_1 > \lambda_2 > \cdots$ are the eigenvalues of L.

We will also determine the relation between the phases. First of all, for the neighbors we have the asymptotic distances

$$\begin{aligned} \beta_{2j-1}^{+} - \beta_{2j}^{+} &= \log(-2\alpha_j^{+}) = \log(4\lambda_j^2) \\ \beta_{n-2j+1}^{-} - \beta_{n-2j+2}^{-} &= \log(-2\alpha_j^{+}) \end{aligned} \tag{7.4}$$

and for the phases of pairs with the same velocities

$$\beta_{2j}^{+} - \beta_{n-2j+2}^{-} = -\sum_{k<j} \log 4(\alpha_{2k}^{+} - \alpha_{2j}^{+})^2 + \sum_{k>j} \log 4(\alpha_{2k}^{+} - \alpha_{2j}^{+})^2. \tag{7.5}$$

Thus the particles undergo a scattering in which the pairs behave as if they interacted pairwise at a time.

The results (7.3), (7.3′), (7.4) are easily derived and we begin with their proof. We recall the differential equation (2.4)

$$\dot{a}_k = a_k(a_{k+1}^2 - a_{k-1}^2), \qquad k = 1, 2, \ldots, n-1$$

with $a_0 = 0 = a_n$ from which we see that

$$\sum_{k=1}^{n-1} a_k^2 = \text{const}$$

along solutions. Thus a_k are bounded. Since

$$\frac{d}{dt} \log(a_1 a_3 \cdots a_{2j-1}) = a_{2j}^2$$

we conclude that

$$\int_0^\infty a_{2j}^2 \, dt < \infty.$$

Since $\dot{a}_{2j}$ is bounded this implies that

$$a_{2j}(t) \to 0 \qquad \text{as} \quad t \to +\infty. \tag{7.6}$$

Thus, the Jacobi matrix $L(t)$, given by (2.1), is asymptotic to a matrix blocked into two by two matrices with eigenvalues $\pm a_{2j-1}(t), j = 1, 2, \ldots, \nu$. Since, on the other hand the eigenvalues λ_k are distinct and independent of t it follows that the limits $a_{2j-1}(t) \to a_{2j-1}(\infty)$ exist and agree with these eigenvalues in some order. From the differential equations

$$\frac{\dot{a}_{2j}}{a_{2j}} = a_{2j+1}^2 - a_{2j-1}^2$$

and from (7.6) it follows that

$$a_{2j+1}^2(\infty) < a_{2j-1}^2(\infty)$$

and thus, if we order the eigenvalues λ_k of L according to (6.1) we conclude

$$a_{2j-1}(t) \to \lambda_j\,, \qquad (j = 1, 2, \ldots, \nu). \tag{7.7}$$

Using the relation

$$4a_k^2 = e^{u_k} = e^{x_k - x_{k+1}} \tag{7.8}$$

we conclude from (7.1), (7.2), (7.7), (7.8) that

$$\dot{x}_{2j}(+\infty) = \dot{x}_{2j-1}(+\infty) = -2\lambda_j^2$$

proving (7.3′) and the first part of (7.3). The other part follows by considering the asymptotic behavior for $t \to -\infty$ analogously.

Moreover, (7.7) and (7.8) implies that

$$x_{2j-1} - x_{2j} \to \log(4\lambda_j^2) \qquad \text{for} \quad t \to +\infty$$

proving the first part of (7.4). The second follows similarly.

It remains to prove (7.5). This will be done by relating the first order differential equations (7.2) to a second order system related to the Toda lattice, for which the scattering problem has been solved [12]. We notice that differentiation of (7.2) yields

$$\begin{aligned}\ddot{x}_k &= -\tfrac{1}{2}(e^{u_k}\dot{u}_k + e^{u_{k-1}}\dot{u}_{k-1}) \\ &= -\tfrac{1}{4}\{e^{u_k}(e^{u_{k+1}} - e^{u_{k-1}}) + e^{u_{k-1}}(e^{u_k} - e^{u_{k-2}})\} \\ &= -\tfrac{1}{4}(e^{x_k - x_{k+2}} - e^{x_{k-2} - x_k})\end{aligned}$$

where we set the undefined exponential terms equal to zero. Thus with

$$\xi_j = x_{2j}; \qquad \tau = t/2 \tag{7.9}$$

we have

$$\frac{d^2\xi_j}{d\tau^2} = e^{\xi_{j-1}-\xi_j} - e^{\xi_j - \xi_{j+1}} = \frac{\partial U}{\partial \xi_j}, \qquad (j = 1, 2,..., \nu) \tag{7.10}$$

where

$$U = \sum_{j=1}^{\nu-1} e^{\xi_j - \xi_{j+1}}.$$

This Hamiltonian system has already been established as an integrable one [13]. For the scattering one has again

$$\xi_j'(+\infty) = \xi'_{\nu+1-j}(-\infty), \qquad j = 1, 2,..., \nu$$

which is consistent with (7.3) as $\xi_j'(\pm\infty) = 2\alpha_{2j}^{\pm}$, and

$$\xi_j(\tau) - \xi_{\nu+1-j}(-\tau) - 2\gamma_j\tau \to \sum_{k\neq j} \delta_{kj} \tag{7.11}$$

where

$$\gamma_j = \xi_j'(+\infty) = 2\alpha_{2j}^+; \qquad \delta_{kj} = \begin{cases} \log(\gamma_k - \gamma_j)^2, & k > j \\ -\log(\gamma_k - \gamma_j)^2, & k < j. \end{cases}$$

With (7.9) the relation (7.11) translates readily into the statement (7.5).

We conclude with a comment on the relation between the differential equation (2.5) by Kac and v. Moerbeke and the equations (7.10) for the the Toda lattice. The first one corresponds to an isospectral deformation of the Jacobi matrix L given by (2.1), with zeros in the diagonal, while the second-order differential equation corresponds to such deformations of such Jacobi matrices with arbitrary diagonal elements (see [4, 12]). To establish the connection between the two we form L^2 which is not any more a tridiagonal matrix, but is similar to one. In fact, with e_α ($\alpha = 1, 2,..., n$) denoting the unit vectors, one finds that L^2 leaves the spaces $E_1 = \text{span}\{e_1, e_3, ..., e_{n-1}\}$ and $E_2 = \text{span}\{e_2, e_4, ..., e_n\}$ invariant and reduces in each of these spaces to a symmetric Jacobi matrix. This explains why the solutions of (2.4) are rationally expressible in terms of $e^{-\lambda_j^2 t}$ while solutions of the corresponding equations for the Toda lattice are rational in $e^{-\lambda_j t}$. This illustrates in a simple example how the operation $L \to L^2$ and more generally $L \to f(L)$ plays a role in these problems.

References

1. V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique, *Ann. Inst. Fourier (Grenoble)* **16** (1966), 319–361.
2. F. Calogero, Solution of the one-dimensional n-body problems with quadratic and/or inversely quadratic pair potentials, *J. Mathematical Phys.* **12** (1971), 419–436.
3. F. Calogero and C. Marchioro, Exact solution of a one-dimensional three-body scattering problem with two-body and/or three-body inverse square potential, *J. Mathematical Phys.* **15** (1974), 1425–1430.
4. H. Flaschka, The Toda lattice, I,[5] *Phys. Rev. B* **9** (1974), 1924–1925.
5. H. Flaschka, The Toda lattice, II, *Progr. Theoret. Phys.* **51** (1974), 703–716.
6. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Korteweg–de Vries equations and generalizations. VI. Methods for exact solutions, *Comm. Pure Appl. Math.* **27** (1974), 97–133.
7. M. Hénon, Integrals of the Toda lattice, *Phys. Rev. B* **9** (1974), 1921–1923.
8. M. Kac and P. van Moerbeke, On an explicitly soluble system of non-linear differential equations related to certain Toda lattices, to appear.

[5] The title reads II, apparently a misprint.

9. M. Kac and P. van Moerbeke, Some probabilistic aspects of scattering theory, to appear.
10. P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* 21 (1968), 467–490.
11. C. Marchioro, Solution of a three-body scattering problem in one dimension, *J. Mathematical Phys.* 11 (1970), 2193–2196.
12. J. Moser, Finitely many mass points on the line under the influence of an exponential potential—An integrable system, to appear in Proc. Battelle Rencontres, Lecture Notes in Physics, Springer, 1975.
13. M. Toda, Waves in nonlinear lattice, *Progr. Theoret. Phys. Suppl.* 45 (1970), 174–200.
14. B. Sutherland, Exact results for a quantum many-body problem in one dimension, II, *Phys. Rev. A* 5 (1972), 1372–1376.
15. V. E. Zakharov and L. D. Faddeev, Korteweg–de Vries equations: A completely integrable Hamiltonian system, *Funkcional Anal. i Priložen* 5 (1971), 18–27.