Physics 210- Fall 2018

Classical and Statistical mechancis

Home Work # 1
Posted on October 5, 2018
Due in Class October 18, 2018

1. *Functional derivatives*
   Consider a functional of a function $\Psi(x)$

$$F[\Psi] = \int_0^L dx\{\frac{a}{2}|\Psi'(x)|^2 + \frac{1}{2}|\Psi(x)|^2 + \frac{g}{4}|\Psi(x)|^4\}.$$

   a) Assuming Ψ is a real function calculate the functional derivative $\frac{\delta F}{\delta \Psi(x)}$. $\ldots[5]$

   b) Assuming Ψ is a complex function calculate the functional derivative $\frac{\delta F}{\delta \Psi^*(x)}$. Here you may assume Ψ, Ψ^* are independent of each other. $\ldots[5]$

   c) In the first case compare the fixed boundary condition (i) $\delta\Psi(0) = 0 = \delta\Psi(L)$ and the periodic boundary condition (ii)$\delta\Psi(0) = \delta\Psi(L) \neq 0$ together with $\Psi'(0) = \Psi'(L)$. $\ldots[5]$

   d) Assuming case (a) and periodic boundary conditions, find the function $\Psi(x)$ which minimizes the functional F at $g = 0, a = 1$ under the constraint of fixed magnitude $\int_0^L dx \Psi(x)|^2 = 1$. (Here you need to set up a differential equation for Ψ and solve it in the case when $g = 0$. This is easy since $g = 0$ reduces it to a linear differential equation.) $\ldots[5]$

   {Comment: This problem gets you going with a set of tricks that are useful in Classical mechanics, and also field theory and quantum mechanics. It is a little beyond what we did in class but hopefully not out of reach. Feel free to ask for help. }

2. *Differential equations and difference equations. Use any convenient software for help with this problem, e.g. Mathematica, Matlab,..*

   Consider the 1-d anharmonic oscillator in dimensionless form

$$H = \frac{p^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}.$$

a) Write the Lagrangian and Hamiltonian equations of motion. . . . [5]

b) Using the discretization $t = j\Delta t,\ j = 0, M - 1$, with a variable M, convert these two equations to difference equations. Solve the two sets of equations from t=0 to t= 10, by iteration for M=10,100,1000 with initial conditions $x(0) = 0; q(0) = \dot{x}(0) = 0.2$ and compare the various solutions. . . . [5]

c) Using a representative M, compare the Hamilton equations solutions with the case $x(0) = -1, q(0) = \dot{x}(0) = 0.2$. . . . [5]

d) Draw the phase portraits of the oscillator, by eliminating t and plotting p versus x by exploring various values of the energy of the oscillator. We expect to see circles surrounding the two points $x = \pm 1$ representing small oscillations around the equilibrium, and a larger set of closed curves surrounding both. These would be separated by a "critical curve" called the separatrix. (This problem has a large fan following in the internet so you should be able to get some help using google scholar.) . . . [5]

3. *Poisson brackets:*

Writing briefly $[] \equiv []_{PB}$, show the properties

a) For any three functions

$$\begin{aligned} [f,g] &= -[g,f] \\ [f+g,h] &= [f,h]+[g,h] \\ [fg,h] &= f[g,h]+[f,h]g \\ [f,[g,h]]+[g,[h,f]]+[h,[f,g]] &= 0, \text{ Jacobi's identity} \end{aligned}$$

{ Comment: These important properties are common to quantum commutators and used frequently, e.g. in part(b) you will need these} . . . [5]

b) Calculate the PB's $[q_i, p_j^3]$, $[Exp[3q_i], p_j^2]$ . . . [5]

4. *Legendre Transforms: General theory and examples*

We may define the Legendre Transform (LT) of any function $F(x)$ as

$$G(y) = \{yx - F(x)\}_{LT} = yx(y) - F(x(y)) \tag{1}$$

where

$$F'(x(y)) = y,$$

i.e. at a given y, we solve for $x(y)$ where the slope of F matches y.

An often added convention: If multiple solutions of $F'(x) = y$ exist, the convention is to choose the solution for which $G''(y) > 0$ i.e. G is a concave-up function of y.

Note: In Eq. (1) we have chosen the sign using the CM convention (used in Classical Mechanics). In Thermodynamics and Stat Mech we will use the SM convention (i.e. the opposite convention), multiply the RHS by -1. With the CM convention the LT of a concave-up function is another concave-up function, while with the SM convention the LT of a concave-up function is another concave-down (or equivalently convex up) function.

a) Calculate $G(y)$ the LT of

$$F(x) = e^{x-1}.$$

. . . [5]

b) Calculate the LT of $G(y)$ and show that we get back $F(x)$. . . . [5]

c) Calculate the LT of

$$F(x) = \frac{x^2}{2} - \frac{x^3}{3}.$$

Show that this leads to two functions $G_1(y)$ and $G_2(y)$. Show that only one of these satisfies the concave-up convention. Graph these functions over a sensible region of x, y . . . [5]

d) Calculate the LT of G_1 and G_2 found above, and show that only one of them recovers the $F(x)$. . . . [5]

5. Considering a relativistic Hamiltonian (1-d)

$$H = \sqrt{p^2c^2 + m^2c^4} + U(q),$$

a) Find Hamilton's equations of motion. . . . [5]

b) Carry out the LT to calculate the Lagrangian. Comment on the form of the kinetic energy in the Lagrangian- is the result what one might have expected? . . . [5]

c) From the Lagrangian calculate the Lagrange equations of motion, and show that they are the same as those in (a). . . . [5]

6. To describe the electromagnetic field interacting with a charged particle in 3-d,we use a Lagrangian

$$L = \frac{m}{2}\dot{\vec{r}}.\dot{\vec{r}} - q_e(\Phi(r) - \frac{1}{c}\dot{\vec{r}}.\vec{A}(\vec{r})) - V(\vec{r}),$$

where q_e is the electron charge, the vector potential $\vec{A}$ and scalar potential Φ lead to EM fields through the usual relations

$$\vec{\nabla} \times \vec{A}(r) = \vec{B}(r),$$

$$\vec{E} = -\vec{\nabla}\Phi(r) - \frac{1}{c}\frac{\partial \vec{A}}{\partial t},$$

and $V(\vec{r})$ is an arbitrary external potential.

a) Using the Legendre transforms, find the Hamiltonian for this problem. ... [5]

b) While L is linear in $\vec{A}$, note that H is quadratic in A. Do you think this quadratic dependence can have observable effects? (A brief answer will suffice). ... [5]

c) From the Lagrange equations of motion show that the force experienced by a particle is

$$m\ddot{\vec{r}} = -\vec{\nabla}V + q_e\{\vec{E} + \frac{\dot{\vec{r}} \times \vec{B}}{c}\}.$$

Note that the second term is the familiar Lorentz force, it is this equation that justifies the choice of the Lagrangian. ... [10]