Physics 210- Fall 2018

Classical and Statistical mechancis

Home Work # 4
Posted on November 29, 2018
Due back on 11 December 2018 (Exam day)

1. In the canonical ensemble we saw that

$$F = -kT \log Z, \quad Z = \sum_n e^{-\beta \varepsilon_n},$$

where ε_n are the eigenvalues of the N particle system. Similarly for the Grand Canonical Ensemble, we saw

$$\Omega = -kT \log Z, \quad Z = \sum_{n,N_n} e^{-\beta(\varepsilon_n - \mu N_n)},$$

a) Show that

$$\bar{N^2} - \bar{N}^2 = \alpha \, \frac{d\bar{N}}{d\mu},$$

with a suitable α. From this show that the mean value $\bar{N}$ is very close to being sharp. (We discussed this in class, I would like you to fill in the details).

b) Show that as $T \to 0$ the variable

$$F - E$$

vanishes, from the property of $E = \bar{\varepsilon}_n$ and the definition of F.

2. a) Using the N spin half problem

$$H = -B \sum_j \sigma_j,$$

with $\sigma_j = \pm 1$, calculate the partition function $Z = \sum_{\sigma_j} e^{-\beta H}$, and the free energy F.

b) Calculate the magnetization $M = -dF/dB$ and susceptibility $\chi = dM/dB/_{B \to 0}$.

c) Calculate the specific heat C as a function of T.

d) Discuss briefly the $T = 0$ limit of C and χ.s

3. Consider the ideal gas of non-interacting particles in the grand canonical ensemble. We can write

$$\Omega(T,\mu,V) = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N(T,V)$$

where Z_N is the canonical partition function

$$Z_N = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{d^3q_i d^3p_i}{h^3}\; e^{-\beta\sum_i \frac{p_i^2}{2m}}.$$

a) Show that

$$Z_N = \frac{1}{N!}\left(\frac{V}{\lambda_{th}^3}\right)^N$$

where $\lambda_{th} = h/\sqrt{2\pi m k_B T}$ is the thermal wavelength.

b) Carry out the sum over N exactly and show that the grand potential

$$\Omega = -k_B T e^{\beta\mu}\frac{V}{\lambda_{th}^3}.$$

c) From Ω calculate expressions for $\bar{N}, P$ in terms of μ, T and V.

d) By inversion find μ as a function of density and T.

4. Consider a set of N quantum harmonic oscillators

$$H = \sum_{i=1}^{N} \hbar\omega(n_i + \frac{1}{2})$$

a) Calculate the partition function Z by summing $e^{-\beta H}$ over all n_i.

b) From this calculate the aveage energy E, and specific heat C as functions of T,N.

c) Find the probability $p(n)$ that a particular oscillator is in its nth quantum level.

5. *This is optional. This is a type of problem that you might encounter in actual experimental physics, and for that reason also in the quals*

An experiment on the heat capacity was performed on a unknown amount of La_2CuO_4, and the resulting data for a large range of T was fit very well to

$$C = \frac{k_B^2 T}{\Delta} \mathrm{sech}^2 \frac{k_B T}{\Delta} \times c_0,$$

where the dimensionless constant $c_0 = 1.025 \times 10^{24}$, and the constant $\Delta = 10$K. Assume that the heat capacity is purely from non-interacting Copper spins (each atom has a spin half), and neglecting phonons, compute the entropy at $T \gg \Delta/k_B$, and from this find the mass of the system.

{Required Data: Atomic weight of Oxygen=16, Cu=63.5,La=139. You will need to argue that each mole containing the Avogadro number of formula units has a known number of copper spin half particles. You will also need to make an educated guess about the entropy of these spins. }