CM 210- Fall 2018. Class notes for solving Legendre Transforms in HW#1
October 31, 2018

Here is a package that does the Legendre transform of most reasonable functions f. I call it LegendreTr[f_,m_]. The parameter m is the number of roots you expect the derivative equation to have. One can alsways start with m=1. One can then try m=2 etc.... We can use it to invert the LT as well.....

```
In[38]:= LegendreTr[f_, m_] := Block[{}, leg = x y - f; der = D[leg, x];
   xsol = x /. Solve[der == 0, x]; Do[g[j] =
     Simplify[(leg /. x → xsol[[j]]), Assumptions -> Element[g[j], Reals]], {j, 1, m}];
   Return[Table[g[j] /. y → x, {j, 1, m}]]]
```

a) Here is one simple example. It is the solution to Problem 4(a,b)

```
In[49]:= f0 = Exp[x - 1]
```

Out[49]= e^{-1+x}

```
In[50]:= lf0 = Simplify[LegendreTr[f0, 1][[1]], Assumptions → Element[x, Reals]]
```

Out[50]= ConditionalExpression$\left[x\left(2\, i\, \pi\, C[1]+\text{Log}[x]\right),\ C[1]\in \text{Integers}\right]$

Here we got the logarithm with its complex
integers. Set C(1)=0 and proceed next.

```
In[53]:= f0 = x Log[x]
```

Out[53]= x Log[x]

```
In[54]:= Simplify[LegendreTr[f0, 1][[1]], Assumptions → Element[x, Reals]]
```

Out[54]= e^{-1+x}

Next we do problem $.c

```
In[55]:= Clear[f0]
```

```
In[56]:= f0 = x^2/2 - x^3/3
```

Out[56]= $\frac{x^2}{2}-\frac{x^3}{3}$

```
In[57]:= lf0 = LegendreTr[x^2/2 - x^3/3, 2]
```

Out[57]= $\left\{-\frac{1}{24}\left(-1+\sqrt{1-4x}\right)\left(-1+\sqrt{1-4x}+8x\right),\ -\frac{1}{24}\left(1+\sqrt{1-4x}\right)\left(1+\sqrt{1-4x}-8x\right)\right\}$

```
In[58]:= G1 = lf0[[1]]
```

Out[58]= $-\frac{1}{24}\left(-1+\sqrt{1-4x}\right)\left(-1+\sqrt{1-4x}+8x\right)$

In[59]:=
```
G2 = lf0[[2]]
```

Out[59]= $$-\frac{1}{24}\left(1+\sqrt{1-4x}\right)\left(1+\sqrt{1-4x}-8x\right)$$

In[65]:=
```
G1curvature = Simplify[D[G1, {x, 2}]]
```

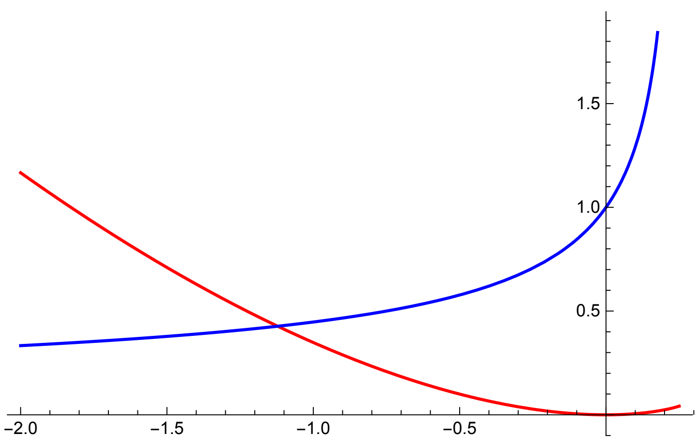
Out[65]= $$\frac{1}{\sqrt{1-4x}}$$

In[66]:=
```
G2curvature = Simplify[D[G2, {x, 2}]]
```

Out[66]= $$-\frac{1}{\sqrt{1-4x}}$$

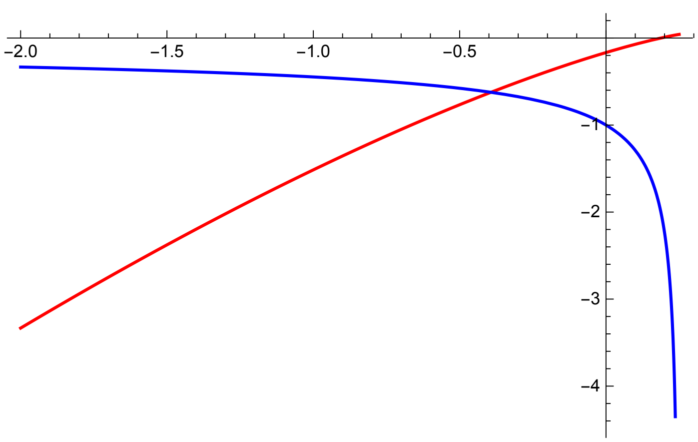
In[67]:=
```
Plot[{G1, G1curvature}, {x, -2, 1/4}, PlotStyle → {Red, Blue}]
```

Out[67]=



In[68]:=
```
Plot[{G2, G2curvature}, {x, -2, 1/4}, PlotStyle → {Red, Blue}]
```

Out[68]=



This shows us that G1 is concave upwards, while G2 is concave downwards

Next we do another LT on G1 and G2!

In[69]:=
```
LG11 = LegendreTr[G1, 1]
```

Out[69]= $$\left\{\frac{1}{24}\left(-24\,(-1+x)\,x^2+\left(-1+\sqrt{(-1+2x)^2}\right)\left(-1+8x-8x^2+\sqrt{(-1+2x)^2}\right)\right)\right\}$$

In[70]:= **LG12 = LegendreTr[G1, 2]**

Part::partw : Part 2 of $\{y-y^2\}$ does not exist. ≫

Part::partw : Part 2 of $\{x-x^2\}$ does not exist. ≫

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General::stop : Further output of Part::partw will be suppressed during this calculation. ≫

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General::stop : Further output of Part::partw will be suppressed during this calculation. ≫

Out[70]=

$$\left\{\frac{1}{24}\left(-24\,(-1+x)\,x^2+\left(-1+\sqrt{(-1+2\,x)^2}\right)\left(-1+8\,x-8\,x^2+\sqrt{(-1+2\,x)^2}\right)\right),\right.$$
$$\left.\frac{1}{12}\left(1-\sqrt{1-4\,\{x-x^2\}[[2]]}\right)+\left(-\frac{1}{2}+x+\frac{1}{3}\sqrt{1-4\,\{x-x^2\}[[2]]}\right)\{x-x^2\}[[2]]\right\}$$

large output | show less | show more | show all | set size limit...

In[71]:= **LG21 = LegendreTr[G2, 1]**

Out[71]=

$$\left\{\frac{1}{24}\left(-24\,(-1+x)\,x^2+\left(1+\sqrt{(-1+2\,x)^2}\right)\left(1-8\,x+8\,x^2+\sqrt{(-1+2\,x)^2}\right)\right)\right\}$$

In[72]:= **LG22 = LegendreTr[G2, 2]**

Part::partw : Part 2 of $\{y-y^2\}$ does not exist. ≫

Part::partw : Part 2 of $\{y-y^2\}$ does not exist. ≫

Part::partw : Part 2 of $\{x-x^2\}$ does not exist. ≫

General::stop : Further output of Part::partw will be suppressed during this calculation. ≫

Out[72]=

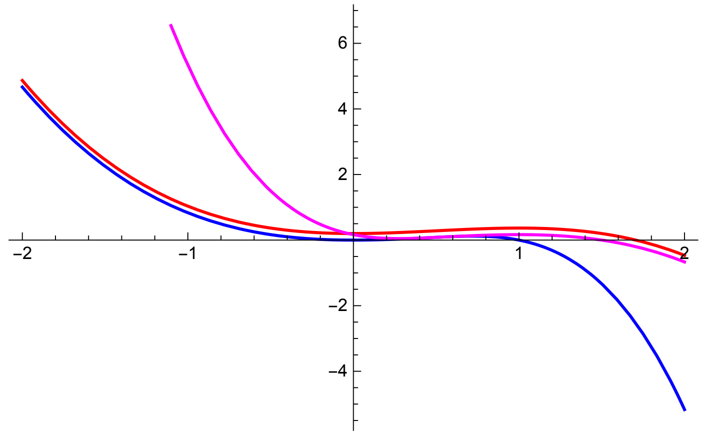
$$\left\{\frac{1}{24}\left(-24\,(-1+x)\,x^2+\left(1+\sqrt{(-1+2\,x)^2}\right)\left(1-8\,x+8\,x^2+\sqrt{(-1+2\,x)^2}\right)\right),\right.$$
$$\left.\frac{1}{12}\left(1+\sqrt{1-4\,\{x-x^2\}[[2]]}\right)+\left(-\frac{1}{2}+x-\frac{1}{3}\sqrt{1-4\,\{x-x^2\}[[2]]}\right)\{x-x^2\}[[2]]\right\}$$

large output | show less | show more | show all | set size limit...

Here we see that only one LT exists for G1 and G2. Let us plot these and compare with original function f0

In[79]:=
```
Plot[{f0 + .2, LG11, LG21}, {x, -2, 2}, PlotStyle → {Red, Blue, Magenta}]
```

Out[79]=



Here we see that the original function f0 (Red) is recovered by piecing together the LT of G1 (Blue) and G2 (Magenta). I have shifted the f0 upwards by a bit to see the differences. This shows the difficulty of solving the LT's of non-concave functions uniquely.