

Physics 210–Fall 2018

Classical and Statistical Mechanics

Finals

100 points, Time 3 hours

December 11, 2018

1. (a) A particle of mass m moves in 2-dimensions without friction on a wire shaped as an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Write down its Lagrangian and the Lagrange equations of motion using a suitable generalized coordinate. (It may be convenient to use $\alpha = (a^2 + b^2)/2$, $\beta = (b^2 - a^2)/2$ instead of a, b .) ... [15]

We can parametrize as

$$x = a \cos(\theta), \quad y = b \sin(\theta),$$

and hence $\dot{x} = -a \sin(\theta)$, $\dot{y} = b \cos(\theta)$. Therefore the Lagrangian

$$L = \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2] = \frac{1}{2}m(\dot{\theta})^2(\alpha + \beta \cos 2\theta).$$

Therefore $\frac{\partial L}{\partial \theta} = m\dot{\theta}(\alpha + \beta \cos 2\theta)$, and $\frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}$.

To obtain the EOM we need

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m\ddot{\theta}(\alpha + \beta \cos 2\theta) - 2\beta m\dot{\theta}^2 \sin 2\theta.$$

Thus the Lagrange EOM $\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$ becomes

$$\ddot{\theta}(\alpha + \beta \cos 2\theta) = \beta \dot{\theta}^2 \sin 2\theta \cdots Eq(a)$$

- (b) Write down the Hamiltonian in the same coordinates as (a). Show that the energy is conserved using the explicit form of the Lagrange equations of motion. ... [10]

We write the momentum from

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}(\alpha + \beta \cos 2\theta) \cdots Eq(b)$$

to eliminate $\dot{\theta}$ in favor of p_θ . Thus

$$H = \frac{1}{2m} \frac{p_\theta^2}{\alpha + \beta \cos 2\theta} \cdots Eq(c)$$

To check the conservation of energy write the EOM $Eq(a)$ as

$$\frac{\ddot{\theta}}{\dot{\theta}} = \dot{\theta} \frac{\beta \sin 2\theta}{(\alpha + \beta \cos 2\theta)} \cdots Eq(a.1)$$

Both sides are exactly integrable. Integrating both sides we get

$$\frac{d}{dt} \log \dot{\theta} = -\frac{1}{2} \frac{d}{dt} \log(\alpha + \beta \cos 2\theta),$$

and hence

$$\frac{d}{dt} \log \left(\dot{\theta}^2 (\alpha + \beta \cos 2\theta) \right) = 0. \cdots Eq(a.2)$$

Using $Eq(b)$ this implies that the time rate of change of $Eq(c)$, i.e. the energy, is zero. **QED**

2. Consider a 1-dimensional elastic continuum with a Lagrangian

$$L[\psi, \dot{\psi}, t] = \frac{1}{2} \int_0^L dx \left[(\dot{\psi}(x, t))^2 - c_0^2 (\partial_x \psi(x, t))^2 - \mu^2 \psi(x, t)^2 \right],$$

where $\psi(x, t)$ is the displacement of the medium at the point x , $\dot{\psi} = \partial_t \psi$, and the action $A = \int_{t_i}^{t_f} L dt$ as usual. Assume periodic boundaries. Find the (Euler) Lagrange equations of motion. Show that these have a wave like solution with frequency

$$\omega_k^2 = c_0^2 k^2 + \mu^2.$$

... [20]

The Lagrange EOM requires us to compute

$$\frac{\delta A}{\delta \dot{\psi}(x, t)} = \dot{\psi}(x, t)$$

$$\frac{\delta A}{\delta \psi(x, t)} = c_0 \partial_x^2 \psi(x, t) - \mu^2 \psi(x, t),$$

these functional derivatives have been calculated by throwing one of the ∂_x to its left using integration by parts, as done in class for the case $\mu = 0$. Hence the EOM satisfied by ψ is

$$\ddot{\psi}(x, t) = c_0 \partial_x^2 \psi(x, t) - \mu^2 \psi(x, t).$$

A wave-like solution of this linear equation clearly exists with

$$\psi(x, t) = A e^{i(kx - \omega_k t)}$$

provided

$$\omega_k^2 = c_0^2 k^2 + \mu^2.$$

3. (a) Consider a gas of particles with energy dispersion $\varepsilon_k = \hbar^2 k^2 / (2m)$ in d -dimensions. For the cases $d = 1, 2$ calculate the density of states

$$n_d(\varepsilon) = \sum_{\vec{k}} \delta(\varepsilon - \varepsilon_k).$$

... [15]

{ Hint: We have derived the density of states for $d = 3$ where the answer is

$$n_3(\varepsilon) = \frac{L^3}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \varepsilon^{\frac{1}{2}}.$$

Use the same method adapted to lower dimensions. }

Let us consider

$$n_1(\varepsilon) = \sum_k \delta(\varepsilon - \varepsilon_k) = \frac{L}{\pi} \int_0^\infty dk \delta(\varepsilon - \varepsilon_k)$$

Changing variables from k to u as

$$u = \hbar^2 k^2 / (2m), \text{ i.e. } k = \frac{\sqrt{2mu}}{\hbar}, \text{ or } dk = \frac{\sqrt{m}}{\hbar\sqrt{2u}} du.$$

Hence

$$n_1(\varepsilon) = \frac{L}{\pi} \int_0^\infty \frac{\sqrt{m}}{\hbar\sqrt{2u}} du \delta(\varepsilon - u) = L \frac{\sqrt{m}}{\pi\hbar\sqrt{2\varepsilon}}.$$

In 2-dimensions we write

$$n_2(\varepsilon) = \sum_k \delta(\varepsilon - \varepsilon_k) = \frac{L^2}{(2\pi)^2} \int dk_x dk_y \delta(\varepsilon - \varepsilon_k)$$

Since k^2 is rotation invariant in 2-d we write a polar decomposition $dk_x dk_y = k dk d\phi$, where $0 \leq \phi \leq 2\pi$. The energy ε_k is independent of ϕ , and can be integrated out. Hence

$$n_2(\varepsilon) = \frac{L^2}{(2\pi)} \int_0^\infty k dk \delta(\varepsilon - \varepsilon_k).$$

We use the same change of variables from k to u as in 1-d, and keep track of the extra factor of k in the integration from change to polar variables. Hence

$$n_2(\varepsilon) = \frac{L^2}{(2\pi)} \frac{m}{\hbar^2} \int_0^\infty du \delta(\varepsilon - u),$$

and hence

$$n_2(\varepsilon) = \frac{L^2}{(2\pi)} \frac{m}{\hbar^2}.$$

(b) Prove the Maxwell relation

$$\left(\frac{\partial p}{\partial N} \right)_V = - \left(\frac{\partial \mu}{\partial V} \right)_N$$

from a suitable thermodynamic potential.

... [10]

Let us consider

$$F = E - TS, \quad dF = -SdT - pdV + \mu dN,$$

so that $\left.\frac{\partial F}{\partial N}\right|_{T,V} = \mu$ and $\left.\frac{\partial F}{\partial V}\right|_{T,N} = -p$. Equating

$$\left.\frac{\partial^2 F}{\partial V \partial N}\right|_T = \left.\frac{\partial^2 F}{\partial N \partial V}\right|_T,$$

we get the desired Maxwell relation.

4. (a) The high T limit of a quantum gas corresponds to

$$e^{\beta(\varepsilon_k - \mu)} \gg 1.$$

Using this approximation in the formula for the number of Fermions or Bosons:

$$\bar{N} = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} \pm 1},$$

calculate the chemical potential

$$\mu = \mu(n, T),$$

for 3-dimensions (where $n = \bar{N}/V$), and for 2-dimensions (where $n = \bar{N}/A$). If possible express the answer in terms of the thermal de-Broglie wavelength $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$. ($V = L^3$, $A = L^2$ are the volume and area respectively). ... [10]

{ Hint: Ignore spin here, and use the density of states $n_3(\varepsilon)$ given above in Problem (3-a). You can use $\int_0^\infty \sqrt{x} e^{-x} dx = \frac{\sqrt{\pi}}{2}$. }

§In 3-dimensions Using the stated approximation we write

$$\bar{N} = e^{\beta\mu} \sum_k e^{-\beta\varepsilon_k} = e^{\beta\mu} \int d\varepsilon n_3(\varepsilon) e^{-\beta\varepsilon} = e^{\beta\mu} \frac{L^3}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty d\varepsilon \varepsilon^{\frac{1}{2}} e^{-\beta\varepsilon}.$$

We carry out the integral by writing $\varepsilon = k_B T x$ so that

$$\int_0^\infty d\varepsilon \varepsilon^{\frac{1}{2}} e^{-\beta\varepsilon} = (k_B T)^{3/2} \int_0^\infty \sqrt{x} e^{-x} dx = \frac{1}{2} (k_B T)^{3/2} \sqrt{\pi}.$$

Plugging in and rearranging we get

$$\frac{\bar{N}}{L^3} = e^{\beta\mu} \left(\frac{\sqrt{2\pi m k_B T}}{h} \right)^3.$$

With density $n = \frac{\bar{N}}{L^3} = \frac{\bar{N}}{V}$ and the thermal de-Broglie wavelength defined above we can write the solution as

$$\mu = k_B T \log(n \lambda_T^3) \dots Eq(d.3)$$

§**In 2-dimensions** again using the stated approximation we write

$$\bar{N} = e^{\beta\mu} \sum_k e^{-\beta\varepsilon_k} = e^{\beta\mu} \int d\varepsilon n_2(\varepsilon) e^{-\beta\varepsilon} = e^{\beta\mu} \frac{L^2}{(2\pi)} \frac{m}{\hbar^2} \int_0^\infty d\varepsilon e^{-\beta\varepsilon} = e^{\beta\mu} \frac{L^2}{(2\pi)} \frac{m}{\hbar^2} (k_B T).$$

Writing $n = \bar{N}/L^2$, this can be rewritten as

$$n = e^{\beta\mu} / \lambda_T^2$$

and hence

$$\mu = k_B T \log n \lambda_T^2 \dots Eq(d.2)$$

This is essentially the same result as in 3-d.

(b) Using the Maxwell relation in problem (3-b), calculate the pressure as a function of N, V, T [5]

From the Maxwell relation and the above calculated μ ,

$$\left(\frac{\partial p}{\partial N} \right)_V = \frac{k_B T}{V}.$$

Integrating this we get the ideal gas equation of state:

$$pV = Nk_B T \dots Eq(e)$$

(c) From (b) and using size scaling $N \rightarrow \lambda N$, $V \rightarrow \lambda V$ applied to the Gibbs free energy $G = E - TS + pV$, calculate G in terms of its natural variables. From G deduce the Arrhenius free energy F in terms of its natural variables. ... [15]

{Recall that the natural variables of G or F are defined from the total differential dG or dF .}

From the definition we gather $G = F + pV$ and hence $dG = -SdT + Vdp + \mu dN$, so that $G = G(p, T, N)$. Size scaling on G gives us

$$\lambda G(p, T, N) = G(p, T, \lambda N).$$

Taking $d/d\lambda$ of both sides and setting $\lambda = 1$ we get

$$G = \mu(p, T)N.$$

This means that if we know μ as a function of p, T we have the G as well. From $Eq(d)$ we know μ as a function of density n and T , so we have to trade these for p, T . From the equation of state $Eq(e)$ we see that $n = p\beta$ and hence we can plug into $Eq(d)$ to get

$$G(p, T, N) = Nk_B T \log(p\beta\lambda_T^3) \cdots Eq(f).$$

To calculate F we note that

$$F = G - pV = Nk_B T \log(p\beta\lambda_T^3) - Nk_B T,$$

by plugging in G . However the natural variables for F are $F(N, V, T)$ so we need to use $p = n\beta$ to get rid of p . This gives the final answer:

$$F = Nk_B T \log \frac{n\lambda_T^3}{e}.$$