Physics 210- Fall 2018

Classical and Statistical mechancis

Home Work # 2
Posted on October 22, 2018
Due in Class October 30, 2018

We will need to use these standard vector identities:

$$\begin{aligned} \vec{A} \times (\vec{B} \times \vec{C}) &= \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}), \quad (I.1) \\ \vec{A}.(\vec{B} \times \vec{C}) &= (\vec{A} \times \vec{B}).\vec{C} \quad (I.2) \\ (\vec{A} \times \vec{B}).(\vec{C} \times \vec{D}) &= (\vec{A}.\vec{C})(\vec{B}.\vec{D}) - (\vec{A}.\vec{D})(\vec{B}.\vec{C}) \quad (I.3) \end{aligned} \tag{1}$$

1. *Generalized Coordinates Example 1*

{Comment: The first two problems are from Landau-Lifshitz Mechanics, where the part (a) of the problems are already solved. We will push ahead a bit more than they do. }

a) Find the generalized coordinates for a coplanar double pendulum. (Solved in LL Problem 1 page 11) . . . [2]

As given in LL, the two angles θ_1, θ_2 serve as generalized coordinates. We can plug in for $x_1, \dot{x_1}$ from the definitions and get the Lagrangian that they derive.

b) Find the equations for the two coordinates ϕ_1, ϕ_2. . . . [5]

We can write the Lagrangian as

$$\begin{aligned} L &= \frac{1}{2}m_T l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2 \dot{\theta}_2^2 + m_T g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \\ &\quad + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \end{aligned}$$

where $m_T = m_1 + m_2$. We take the Lagrange equations in a straightforward way to get

$$m_T l_1^2(\ddot{\theta}_1 + \frac{g}{l_1}\sin\theta_1) + m_2 l_1 l_2 \left\{ \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \right\} = 0$$
$$m_2 l_2^2(\ddot{\theta}_2 + \frac{g}{l_2}\sin\theta_2) + m_2 l_1 l_2 \left\{ \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \right\} = 0$$

The second equation can be written down from the first one by noticing the symmetry of the first 4 terms of the Lagrangian in exchanging $(m_T, l_1, \theta_1) \leftrightarrow (m_2, l_2, \theta_2)$.

c) Comment on how you would solve these problems. (If you actually can solve them on a computer that would earn some extra credit) . . . [3]

The term in curly brackets is the coupling between the two angles. It makes the problem quite messy and the solution is "chaotic", i.e. two nearly equal initial conditions lead to solutions that diverge very far from each other. Numerical solutions of such problems are very tricky and a great deal of expertise is needed to get accurate solutions.

2. *Generalized Coordinates Example 2*

a) Find the generalized coordinates for a simple pendulum of mass m_2 moving in the $x-y$ plane supported with a mass m_1 that is constrained to lie on a horizontal line along x axis. (Solved in LL Problem 2 page 11. See the figure in the book). . . . [2]

b) Write down the equations for the x and ϕ variables. . . . [5]

c) Comment on how you would solve these two equations. (If you actually can solve them on a computer that would earn some extra credit) . . . [3]

3. *Lenz vector problems*

a) For the gravitational problem $V(r) = -\frac{k}{r}$, we wrote down the Lenz vector $\vec{A} = \dot{\vec{r}} \times \vec{L} - k\frac{\vec{r}}{r}$. Using the equation of motion for $\vec{r}$ show that $\vec{A}$ is conserved. . . . [5]

The EOM in vector notation is

$$m\ddot{\vec{r}} = -k\frac{\vec{r}}{r^3}.$$

Taking the derivative of $\vec{A}$ we get

$$\dot{\vec{A}} = \ddot{\vec{r}} \times \vec{L} - k\frac{\dot{\vec{r}}}{r} + k\frac{\vec{r}(\vec{r}.\dot{\vec{r}})}{r^3}$$

or using the EOM and $\vec{L} = m\vec{r} \times \dot{\vec{r}}$

$$\dot{\vec{A}} = -k\frac{\vec{r} \times (\vec{r} \times \dot{\vec{r}})}{r^3} - k\frac{\dot{\vec{r}}}{r} + k\frac{\vec{r}(\vec{r}.\dot{\vec{r}})}{r^3}.$$

Using vector identity (I.1) this term vanishes identically. *QED*

b) From the above equation of motion show that $\vec{A}.\vec{r} - kr = \frac{L_z^2}{m}$, i.e is the equation of an ellipse. What is the eccentricity e in terms of A? . . . [5]

We see from the definition of $\vec{A}$ that

$$\vec{r}.\vec{A} = -kr + \vec{r}.(\dot{\vec{r}} \times \vec{L}).$$

We can use identity I.2 and write $\dot{\vec{r}} = \frac{\vec{p}}{m}$ and use $\vec{r} \times \vec{p} = \vec{L}$ to write

$$\vec{r}.\vec{A} = -kr + \frac{1}{m}\vec{L}.\vec{L}$$

This is the required answer with the vector $\vec{L}$ chosen along the z axis.

We use the standard form of the ellipse in polar coordinates as

$$\frac{p}{r} = 1 - e\cos(\theta),$$

where e is the eccentricity and p is the other parameter of the ellipse. In terms of these parameters the semi-major and semi-minor radii are

$$a = \frac{p}{1-e^2}, \;\; b = \frac{p}{\sqrt{1-e^2}}.$$

Let us note a fact that is useful below. We can rewrite the equation for the ellipse from $cos\theta = \frac{1}{e}(1 - \frac{p}{r})$ and taking the derivative w.r.t. r we get

$$d\theta = \frac{dr}{r^2\sqrt{-\frac{1}{r^2} + \frac{e^2-1}{p^2} + \frac{2}{pr}}}. \text{ Differential form of an ellipse}$$

If we now align the x axis along $\vec{A}$ then $vecr.\vec{A} = Ar\cos\phi$, and hence we can rewrite the above as

$$\frac{L^2}{mk\,r} = 1 - \frac{A}{k}\cos\phi,$$

and hence the eccentricity can be written as:

$$e = \frac{A}{k}.$$

c) Show that that $e^2 = 1 + 2EL_z^2/(mk^2)$ when expressed in terms of the energy E, and thus relate $|A|$ to E. ...[10]

We solve the Kepler problem in spherical coordinates as in class to obtain

$$\dot{r} = \sqrt{2/m}\sqrt{E - \frac{L_z^2}{2mr^2} - \frac{k}{r}},$$

where $r = |\vec{r}|$ and $E = -|E|$ is the energy of the bound state. We can rewrite this in terms of the azimuthal angle ϕ by using

$$mr^2\dot{\phi} = L_z, \quad dt = \frac{mr^2}{L_z}d\phi$$

and hence using this to rewrite above as

$$\frac{dr}{d\phi} = r^2\sqrt{\frac{2m}{L_z^2}}\sqrt{E - \frac{L_z^2}{2mr^2} - \frac{k}{r}},$$

or

$$d\phi = \frac{dr}{r^2\sqrt{-\frac{1}{r^2} + \frac{2mk}{L_z^2 r} + \frac{2mE}{L_z^2}}}.$$

This is the equation of the ellipse in differential form as we saw above. We can then read off

$$p = \frac{L_z^2}{mk}, \quad \frac{e^2 - 1}{p^2} = \frac{2mE}{L_z^2},$$

and hence

$$e = \sqrt{1 + \frac{2L_z^2 E}{mk^2}}.$$

We saw earlier that $e = A/k$ and hence comparing we get

$$A = \sqrt{k^2 + \frac{2L_z^2 E}{m}}.$$

4. *Central field problem*

Assuming that the central potential is given by $V(r) = -\frac{k}{r^\sigma}$, with $\sigma = 1, 1.5, 2$ and choosing suitable initial conditions and an illustrative value of the conserved energy and (non-zero) angular momentum:

a) Compute and plot $r(t)$ versus t for a sufficiently large range of times t,

b) Compute and plot $\phi(t)$ versus t using the above solution (from $mr^2\dot{\phi} = L_z$).

c) Eliminate t and plot $r(t)$ versus ϕ to illustrate that the orbits are closed in the case of $\sigma = 1$ and not otherwise. In other cases show that the $r(t) - \phi(t)$ curves are space filling. ...[10]

5. *Velocity dependent forces and energy conservation*

We generalize Lagrange's equations to a more general form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} + Q[q, \dot{q}]$$

The case of physical interest in viscous damping has

$$Q = -k\dot{q}$$

with $k > 0$

a) Show that the equation of motion exhibits damping i.e. decay at long times by solving exactly the (simple) examples of $V = 0, \frac{kq^2}{2}$. (Here V is the potential energy in L). ...[5]

The Lagrange EOM for a free particle follows by plugging in $L = \frac{1}{2}m\dot{q}^2$ as

$$m\ddot{q} = -k\dot{q}.$$

By inspection we can write the solution as

$$q = a_0 + a_1 e^{-kt/m},$$

where a_0, a_1 are arbitrary constants. The solution clearly decays to zero as $t \to \infty$.

For the Harmonic oscillator we write $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}aq^2$, with a spring constant a. The EOM is clearly

$$m\ddot{q} + k\dot{q} + aq = 0.$$

It is linear and hence we can solve it easily using $q = e^{i\omega t}$ form. This gives

$$\omega^2 - i\omega\frac{k}{m} - \frac{a}{m} = 0.$$

The two roots (with $\frac{a}{m} - (\frac{k}{2m})^2 > 0$) are

$$\omega = \omega_\pm = i\frac{k}{2m} \pm \sqrt{\frac{a}{m} - (\frac{k}{2m})^2}.$$

The general solution is

$$q(t) = A_+ e^{i\omega_+ t} + A_- e^{i\omega_- t}.$$

This decays in time for large positive t since the imaginary part of the $\omega_\pm$ is positive.

b) With energy $E \equiv \frac{m\dot{q}^2}{2} + V(q)$, show that its rate of change is negative, i.e. $dE/dt < 0$, due to damping. What does this mean physically? ...[5]

With $L = \frac{1}{2}m\dot{a}^2 - V(q)$ the EOM in this problem is given as

$$m\ddot{q} = -k\dot{q} - \frac{\partial V}{\partial q}.$$

We can follow the procedure we used for showing that energy is constant for the undamped case, and add the damping term at the end. Let us construct the object

$$\frac{d}{dt}\{m\dot{q}^2 - L\} = 2m\dot{q}\ddot{q} - m\dot{q}\ddot{q} + \dot{q}\frac{\partial V}{\partial q}.$$

Simplifying and substituting for $m\ddot{q}$ from the EOM, we get

$$\frac{d}{dt}\{\frac{1}{2}m\dot{q}^2 + V\} = -k\dot{q}^2$$

. This can be interpreted as

$$\frac{dE}{dt} = -k\dot{q}^2 < 0,$$

where the inequality follows from the squaring of $\dot{q}$. This means physically that the energy decreases in time for any damped system. This is reasonable since damping implies friction, which dissipates energy.