Physics 210- Fall 2018

Classical and Statistical mechancis

Solution to Home Work # 3

Solution Posted on November 29, 2018

1. *Canonical Transformations Example 1*

a) From the theory of canonical transformations calculate the transformation generated by $F_1(q, Q) = Q/q$ of the free particle problem

$$H = p^2/(2m).$$

. . . [10]

From the "theoretical" equations $p = \partial F_1/\partial q$ and $P = -\partial F_1/\partial Q$ obtained in class, we see that

$$p = -Q/q^2, \;\; P = -1/q.$$

Therefore we can solve for the old in terms of the new variables,

$$q = -1/P, \quad p = -QP^2$$

or the inverses,

$$P = -1/q, \quad Q = -pq^2.$$

Hence in the new variables the free particle Hamiltonian $H = p^2/(2m)$ becomes:

$$H = P^4Q^2/(2m).$$

b) Find the Hamiltonian equations of motion in the new representation, and solve them exactly. . . . [10]

The new Hamiltonian EOM read

$$\dot{Q} = \{Q, H\}, \;\; \dot{P} = \{P, H\}$$

where the brackets are the Poisson brackets. Thus the strange looking EOM are now

$$\dot{Q} = 2Q^2P^3/m, \ \ \dot{P} = -QP^4/m.$$

To solve them we observe that

$$\dot{Q}P + 2Q\dot{P} = 0,$$

and hence

$$dQ/Q + 2dP/P = 0$$

so that

$$QP^2 = A,$$

where A is some constant. This means that $dP/dt = -AP^2/m$ which can be solved easily by separating terms so that

$$dP/P^2 = -a/mdt$$

so integrating

$$1/P = B + At/m$$

and

$$Q = A/P^2 = A(B + At/m)^2.$$

We see that these correspond to the usual solution of the free particle problem $p = A$ where A=const, and $x = x_0 + tA$.

2. *Canonical Transformations Example 2*

a) Show that a canonical transformation from q, p to any required $Q \equiv Q(q)$ (i.e. a function of q only) can be generated by the generator $F_2(q, P)$. ...[5]

We can choose $F_2(q, P) = PQ(q)$ where $Q(q)$ is an arbitrary function of q. From the theory of transformations, this implies two equations

$$p = \partial F_2/\partial q, \ \ Q = \partial F_2/\partial P = Q(q).$$

With these two equations we can fully invert and express q, p in terms of Q, P. Since Q is only a function of q and not of p, this transformation is called a *contact transformation.*

b) Find the $F_2(q, P)$ necessary to make the transformation in 2 dimensions from $\vec{r} = \{x, y\}$ to the standard polar coordinates r, θ . ...[5]

We want to transform from x, y to r, θ using

$$x = r\cos(\theta),\ y = r\sin(\theta)$$

Hence (q, p) variables are the pair (x, p_x) and (y, p_y) and $Q_1 = r,\ \ Q_2 = \theta$. The theory helps us to compute P_1, P_2 automatically so that the new set is also canonical.

Writing

$$F_2(q, P) = P_1 r + P_2 \theta,$$

with

$$Q_1 = r = \sqrt{x^2 + y^2},\ \text{and}\ \ Q_2 = \theta = \arctan(y/x).$$

c) Using (b) find the full transformation from $\vec{q}, \vec{p}$ to the new canonical momenta and coordinates. ... [5]

We now compute

$$p_x = \partial F_2/\partial x = \frac{x}{r}P_1 - \frac{y}{r^2}P_2,$$

$$p_y = \partial F_2/\partial y = \frac{y}{r}P_1 + \frac{x}{r^2}P_2.$$

Hence the inversion is easy

$$P_1 = p_x\frac{x}{r} + p_y\frac{y}{r},$$

$$P_2 = xp_y - yp_x.$$

Note that P_2 is the angular momentum L^z, as one expects.

d) Verify that the new coordinates satisfy the canonical algebra by computing the 4 poisson brackets $\{Q_i, P_j\}_{q,p}$. ... [5]

For this problem we use the definition of the Poisson brackets

$$\{A, B\} = (\partial A/\partial x)\,(\partial B/\partial p_x) - (\partial A/\partial p_x)\,(\partial B/\partial x) + (x \leftrightarrow y).$$

Working through the partial derivatives, we can check the quoted result. We should note that Q_1, Q_2 only depend on x, y and not on p_x, p_y and hence half the terms in the Poisson brackets are identically zero.

3. *Action problem*

a) For the simple harmonic oscillator

$$H = p^2/(2m) + kq^2/2,$$

calculate the action

$$J(E) = \oint pdq,$$

by integrating over a complete cycle. From the derivative with respect to energy, calculate the time period. . . . [5]

There are many ways of calculating the $J(E)$. The simplest is to use Greens theorem which relates it to the area of the surface in (p,q) plane, with $p^2/(2m) + kq^2/2 \leq E$. A more algebraic method is useful since it generalizes to the next problem too. We solve $p^2/(2m) + kq^2/2 = E$ for p in terms of q, E and find two roots

$$p_\pm = \pm\sqrt{mk}\sqrt{q_m^2 - q^2},$$

where

$$q_m = \sqrt{2E/k}.$$

The two roots correspond to the two signs of the velocity, rightward or leftwards. We need to use both of them for the closed path integral. By a simple argument

$$J(E) = \int_{-q_m}^{q_m} dq\ p_+ + \int_{q_m}^{-q_m} dq\ p_- \ = 2\int_{-q_m}^{q_m} dq\ p_+$$

where the closed path is being traversed in a clockwise sense. Writing $q = q_m \cos(\theta)$ the integral is written as

$$J(E) = -2\sqrt{mk}\ q_m^2 \ \int_0^\pi sin^2\theta\ d\theta == -2\pi\sqrt{m/k}\ E.$$

We may use the formula

$$T(E) = dJ(E)/dE = -2\pi\sqrt{m/k}.$$

The sign is ignored since it switches with the sense in which we go around the orbit.This gives the time period, which is independent of E for the harmonic oscillator.

b) Do the same calculation for the quartic oscillator

$$H = p^2/(2m) + kq^4/4,$$

where you can use scaling to get the energy dependence of the action, and ignore (i.e. leave undetermined) the fairly cumbersome integral, which is dimensionless and hence less important. ... [5]

This problem can be done in a very similar way as above, we need to redefine

$$q_m = (4E/k)^{\frac{1}{4}},$$

so that

$$p_\pm = \pm\sqrt{mk/2}\sqrt{(q_m^4 - q^4)}.$$

Repeating the steps in part (a), we get

$$J(E) = 2\sqrt{mk/2}\int_{-q_m}^{q_m} dq\ \sqrt{(q_m^4 - q^4)}.$$

We next scale $q = q_m \cos\theta$ so that

$$J(E) = 2\sqrt{mk/2}q_m^3 \int_0^\pi d\theta\ \sin\theta\ \sqrt{1-\sin^4\theta}.$$

We can write this as

$$J(E) = AE^{3/4},$$

where A is independent of the energy E. Taking the derivative

$$dJ/dE = T(E) = 3A/4E^{-1/4}.$$

c) Show that time period can be written as

$$T(E) = \int\int dp\, dq\ \delta(H-E),$$

by differentiating the formula for $J(E)$ and evaluate this integral for the Harmonic oscillator directly to confirm the result of (a). ... [10]

As discussed in part (a) we can write

$$J(E) = \int\int dp\ dq\ \Theta(E - p^2/(2m) - kq^2/2).$$

We use

$$T(E) = dJ(E)/dE = \int\int dp\ dq\ \delta(E - p^2/(2m) - kq^2/2),$$

where we took the derivative inside the integral sign and used the hint $d\Theta(x)/dx = \delta(x)$.

{ Hint: This problem requires you to use the familiar formulat $\delta(x) = \frac{d}{dx}\Theta(x)$ where Θ is the Heaviside step function. }

4. *Thermodynamics and partial derivatives Example 1*

a) We worked out a few examples of thermodynamic potentials,

$$\begin{aligned} dE &= T\,dS - p\,dV + \mu\,dN \\ dF &= -S\,dT - p\,dV + \mu\,dN \\ d\Omega &= -S\,dT - p\,dV - N\,d\mu \end{aligned}$$

Using the standard conditions for exact differentials, this leads to the Maxwell relations. For example from the first equation we read

$$\frac{\partial^2 E}{\partial S \partial V} = \frac{\partial^2 E}{\partial V \partial S}$$

and hence

$$-\frac{\partial p}{\partial S}|_{V,N} = \frac{\partial T}{\partial V}|_{S,N}.$$

This is an example of a Maxwell relation. It is infact rather useless since we did not choose the potential strategically. We get more useful ones by rewriting the first equation by moving S to one side and everything else to the other. Doing this, write down the 3 Maxwell relations from the first equation. . . . [10]

b) Similarly write down the 4 Maxwell relations from the second and third potentials by dropping the number variation. . . . [10]

5. *Thermodynamics and partial derivatives Example 2*

Using the tricks in Landau Lifshitz Stat Mech (Pages 49-51 - scan in the website) show

a)

$$\frac{\partial C_p}{\partial P}|_T = -T\frac{\partial^2 V}{\partial T^2}|_P$$

This is Eq 16.2 of LL. . . . [10]

b) Show that (16.6)

$$\frac{\partial E}{\partial P}|_T = -T\frac{\partial V}{\partial T}|_P - P\frac{\partial V}{\partial P}|_T$$

. . . [5]

c)Show that (16.8-1) . . . [5]

$$\frac{\partial E}{\partial T}|_P = C_p - P\frac{\partial V}{\partial T}|_P$$

6. *No submission required but verify these important identities for Jacobians*

   Page 51 LL, (I), (II), (III), (IV), (V).

   {We will use these in the next few classes }