Class of 14th Nov 2017

Then, comparing this expression with $\mathscr{D}(\tau > 0)$, we arrive at the conclusion that the temperature Green's function for phonons is an even function of τ:

$$\mathscr{D}(\tau) = \mathscr{D}(-\tau). \tag{11.9}$$

This result is valid for the Green's function of any real field.

11.2 Temperature Green's functions for free particles. In a perturbation theory based on the diagram technique, free-particle Green's functions play an important role. In the absence of interactions, the statistical average in (11.1) can be carried out over the states of the individual particles separately. The energy levels E_n of the system can be written as a sum over the energies of the individual particles (in states with a given momentum $\mathbf{p}$ and spin projection α), and the same is true of the thermodynamic potential Ω:

$$E_n^{(0)} = \sum_{\mathbf{p},\alpha} n_{\mathbf{p}\alpha}\varepsilon_0(\mathbf{p}), \qquad \Omega_0 = \sum_{\mathbf{p},\alpha} \Omega_{\mathbf{p}\alpha}^{(0)}.$$

In the case of Fermi statistics, the occupation numbers of the states can only take the values 0 and 1, because of the Pauli principle.

To calculate the free-particle Green's functions, it is most convenient to use the definition (11.1). Substituting into (11.1) the Fourier expansions of the operators ψ and ψ^+

$$\psi_\alpha(\mathbf{r}_1) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}_1} a_{\mathbf{p}_1\alpha}e^{i\mathbf{p}_1\cdot\mathbf{r}_1}, \qquad \psi_\beta^+(\mathbf{r}_2) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}_2} a_{\mathbf{p}_2\beta}^+e^{-i\mathbf{p}_2\cdot\mathbf{r}_2},$$

we obtain

$$\mathscr{G}_{\alpha\beta}^{(0)}(\tau > 0) = -\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2} e^{i(\mathbf{p}_1\cdot\mathbf{r}_1 - \mathbf{p}_2\cdot\mathbf{r}_2)} \times \operatorname{Sp}\{e^{(\Omega_0+\mu\hat{N}-\hat{H}_0)/T}e^{(\hat{H}_0-\mu\hat{N})\tau}a_{\mathbf{p}_1\alpha}e^{-(\hat{H}_0-\mu\hat{N})\tau}a_{\mathbf{p}_2\beta}^+\}.$$

Moreover, bearing in mind that in the interaction representation the Hamiltonian $\hat{H}$ has the form

$$\hat{H}_0 = \sum_{\mathbf{p},\alpha}\hat{n}_{\mathbf{p}\alpha}\varepsilon_0(\mathbf{p}), \qquad \hat{N} = \sum_{\mathbf{p},\alpha}\hat{n}_{\mathbf{p}\alpha},$$

we can easily verify the identities

$$\begin{aligned} e^{(\hat{H}_0-\mu\hat{N})\tau}a_{\mathbf{p}\alpha}e^{-(\hat{H}_0-\mu\hat{N})\tau} &= a_{\mathbf{p}\alpha}e^{-[\varepsilon_0(\mathbf{p})-\mu]\tau}, \\ e^{(\hat{H}_0-\mu\hat{N})\tau}a_{\mathbf{p}\alpha}^+e^{-(\hat{H}_0-\mu\hat{N})\tau} &= a_{\mathbf{p}\alpha}e^{[\varepsilon_0(\mathbf{p})-\mu\hat{N}]\tau} \end{aligned} \tag{11.10}$$

(it is sufficient to calculate the only nonzero matrix element in the left and right-hand sides). Thus we have

$$\mathscr{G}^{(0)}(\tau > 0) = -\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2} e^{i(\mathbf{p}_1\cdot\mathbf{r}_1-\mathbf{p}_2\cdot\mathbf{r}_2)-[\varepsilon_0(\mathbf{p}_1)-\mu]\tau}\operatorname{Sp}\{e^{(\Omega_0+\mu\hat{N}-\hat{H}_0)/T}a_{\mathbf{p}_1\alpha}a_{\mathbf{p}_2\beta}^+\}.$$

↗

Recall $g^{(0)}(\tau) = -\theta(\tau)\langle a(\tau)a^+\rangle + \theta(-\tau)\langle a^+ a(\tau)\rangle$

Since the product $a_{\mathbf{p}_1\alpha}a_{\mathbf{p}_2\beta}^+$ has nonzero diagonal matrix elements only for $\mathbf{p}_1 = \mathbf{p}_2$, $\alpha = \beta$, it follows that

$$\mathscr{G}_{\alpha\beta}^{(0)}(\mathbf{r}_1 - \mathbf{r}_2, \tau > 0) = -\delta_{\alpha\beta}\frac{1}{V}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)-[\varepsilon_0(\mathbf{p})-\mu]\tau}\langle a_{\mathbf{p}\alpha}a_{\mathbf{p}\alpha}^+\rangle.$$

The quantity $\langle a_{\mathbf{p}\alpha}a_{\mathbf{p}\alpha}^+\rangle$ can be expressed in terms of the equilibrium occupation numbers $n(\mathbf{p})$, which depend on the temperature and the chemical potential. For fermions we have

$$\langle a_{\mathbf{p}\alpha}a_{\mathbf{p}\alpha}^+\rangle = 1 - n(\mathbf{p}), \qquad n(\mathbf{p}) = \{e^{[\varepsilon_0(\mathbf{p})-\mu]/T} + 1\}^{-1}, \tag{11.11}$$

and for bosons

$$\langle a_{\mathbf{p}\alpha}a_{\mathbf{p}\alpha}^+\rangle = 1 + n(\mathbf{p}), \qquad n(\mathbf{p}) = \{e^{[\varepsilon_0(\mathbf{p})-\mu]/T} - 1\}^{-1}. \tag{11.12}$$

We now let the volume V of the system approach infinity, going over from summation over the momenta to integration, in the usual way. The final result is

$$\mathscr{G}_{\alpha\beta}^{(0)}(\mathbf{r}, \tau > 0) = -\delta_{\alpha\beta}\frac{1}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{r}-[\varepsilon_0(\mathbf{p})-\mu]\tau}[1 \mp n(\mathbf{p})], \tag{11.13a}$$

||

where the upper sign corresponds to fermions, and the lower sign to bosons. ←
To calculate $\mathscr{G}^{(0)}$ for $\tau < 0$, it is simplest to use the relation (11.8):

$$\begin{aligned} \mathscr{G}_{\alpha\beta}^{(0)}(\mathbf{r}, \tau < 0) &= \mp\,\mathscr{G}_{\alpha\beta}^{(0)}\left(\mathbf{r}, \tau + \frac{1}{T}\right) \\ &= \pm\,\delta_{\alpha\beta}\frac{1}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{r}-[\varepsilon_0(\mathbf{p})-\mu]\tau}n(\mathbf{p}). \end{aligned} \tag{11.13b}$$

The Green's function for free phonons is calculated similarly. Substituting into (11.2) the Fourier expansion of the operator $\varphi(\mathbf{r})$

$$\varphi(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\sqrt{\frac{\omega_0(\mathbf{k})}{2}}\,(b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + b_{\mathbf{k}}^+e^{-i\mathbf{k}\cdot\mathbf{r}}),$$

where $\omega_0(\mathbf{k})$ is the energy of the phonon, we find after making the appropriate calculations that

$$\begin{aligned} \mathscr{D}^{(0)}(\mathbf{r}, \tau) = -\frac{1}{2(2\pi)^3}\int d\mathbf{k}\,\omega_0(\mathbf{k})\{&[N(\mathbf{k}) + 1]e^{i\mathbf{k}\cdot\mathbf{r}-\omega_0(\mathbf{k})|\tau|} \\ &+ N(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}+\omega_0(\mathbf{k})|\tau|}\}, \end{aligned} \tag{11.14}$$

←

where

$$N(\mathbf{k}) = [e^{\omega_0(\mathbf{k})/T} - 1]^{-1}.$$

According to (11.9), $\mathscr{D}^{(0)}$ is an even function of τ.

12. Perturbation Theory

12.1. The interaction representation. If the particles making up the system are not free, then in the expression for the temperature Green's function (11.1), we can go over to a special interaction representation, like the interaction representation of quantum field theory [Matsubara (M1)]. To this end, we introduce a matrix $\mathscr{S}(\tau)$, where $0 < \tau < 1/T$, analogous to the S-matrix of field theory and defined by the relations

$$\left.\begin{aligned} e^{-(\hat{H}-\mu\hat{N})\tau} &= e^{-(\hat{H}_0-\mu\hat{N})\tau}\mathscr{S}(\tau), \\ e^{(\hat{H}-\mu\hat{N})\tau} &= \mathscr{S}^{-1}(\tau)e^{(\hat{H}_0-\mu\hat{N})\tau}. \end{aligned}\right\} \text{Crucial Result} \qquad (12.1)$$

Moreover, we introduce the particle-field operators

$$\begin{aligned} \psi(\mathbf{r}, \tau) &= e^{(\hat{H}_0-\mu\hat{N})\tau}\psi(\mathbf{r})e^{-(\hat{H}_0-\mu\hat{N})\tau}, \\ \bar{\psi}(\mathbf{r}, \tau) &= e^{(\hat{H}_0-\mu\hat{N})\tau}\psi^+(\mathbf{r})e^{-(\hat{H}_0-\mu\hat{N})\tau} \end{aligned} \qquad (12.2)$$

in the interaction representation, which for $\hat{H} = \hat{H}_0$ coincide with the Heisenberg operators mentioned in Sec. 11. Other operators in the interaction representation are introduced by analogy with (12.2). In particular, we have

$$\begin{aligned} \hat{H}(\tau) &= e^{(\hat{H}_0-\mu\hat{N})\tau}\hat{H}e^{-(\hat{H}_0-\mu\hat{N})\tau}, \\ \hat{H}_{\text{int}}(\tau) &= e^{(\hat{H}_0-\mu\hat{N})\tau}\hat{H}_{\text{int}}e^{-(\hat{H}_0-\mu\hat{N})\tau}. \end{aligned}$$

This definition implies that the operators $\hat{H}(\tau)$, $\hat{H}_{\text{int}}(\tau)$ are obtained from $\hat{H}$, $\hat{H}_{\text{int}}$ by replacing $\psi(\mathbf{r})$, $\psi^+(\mathbf{r})$ by $\psi(\mathbf{r}, \tau)$, $\bar{\psi}(\mathbf{r}, \tau)$, respectively. Moreover, we note that $\hat{H}_0(\tau)$ and $\hat{N}(\tau)$ actually do not depend on τ (the free-particle Hamiltonian commutes with the operator $\hat{N}$):

$$\begin{aligned} \hat{H}_0(\tau) &= e^{(\hat{H}_0-\mu\hat{N})\tau}\hat{H}_0e^{-(\hat{H}_0-\mu\hat{N})\tau} = \hat{H}_0, \\ \hat{N}(\tau) &= e^{(\hat{H}_0-\mu\hat{N})\tau}\hat{N}e^{-(\hat{H}_0-\mu\hat{N})\tau} = \hat{N}. \end{aligned}$$

The matrix $\mathscr{S}(\tau)$ satisfies a simple equation, which differs from the corresponding equation (6.27) for the S-matrix by the substitution $t \to -i\tau$. However, we now give a fresh derivation of this equation. Differentiating the first of the equations (12.1) with respect to τ, we obtain

$$-(\hat{H} - \mu\hat{N})e^{-(\hat{H}-\mu\hat{N})\tau} = e^{-(\hat{H}_0-\mu\hat{N})\tau}\frac{\partial\mathscr{S}(\tau)}{\partial\tau} - (\hat{H}_0 - \mu\hat{N})e^{-(\hat{H}_0-\mu\hat{N})\tau}\mathscr{S}(\tau).$$

Multiplying both sides of this equation by $e^{(\hat{H}_0-\mu\hat{N})\tau}$, we have

$$\frac{\partial\mathscr{S}(\tau)}{\partial\tau} = -\hat{H}_{\text{int}}(\tau)\mathscr{S}(\tau). \qquad (12.3)$$

The solution of equation (12.3) satisfying $\mathscr{S}(0) = 1$, which follows from the definition of $\mathscr{S}$, has the form

$$\mathscr{S}(\tau) = T_\tau \exp\left\{-\int_0^\tau \hat{H}_{\text{int}}(\tau')\,d\tau'\right\}. \longleftarrow \text{Dyson's solution for } \mathscr{S} \qquad (12.4)$$

As already noted, the symbol T_τ means that all operators must be arranged from left to right in order of decreasing τ. We can easily verify the validity of (12.4) by direct differentiation, taking account of the operation T_τ just mentioned.

Besides $\mathscr{S}(\tau)$, we consider another matrix $\mathscr{S}(\tau_1, \tau_2)$, where $\tau_1 > \tau_2$ is defined by

$$\mathscr{S}(\tau_1, \tau_2) = T_\tau \exp\left\{-\int_{\tau_2}^{\tau_1} \hat{H}_{\text{int}}(\tau')\,d\tau'\right\},$$

$$\boxed{\mathscr{S}(\tau) = \mathscr{S}(\tau, 0).}$$

Obviously, $\mathscr{S}(\tau_1, \tau_2)$ has the properties , Group Property

$$\begin{aligned} \mathscr{S}(\tau_1, \tau_3) &= \mathscr{S}(\tau_1, \tau_2)\mathscr{S}(\tau_2, \tau_3) \quad \text{for} \quad \tau_1 > \tau_2 > \tau_3, \\ \mathscr{S}(\tau_1, \tau_2) &= \mathscr{S}(\tau_1)\mathscr{S}^{-1}(\tau_2) \qquad \text{for} \quad \tau_1 > \tau_2. \end{aligned} \qquad (12.5)$$

We now go over to the interaction representation in formula (11.1) for the Green's function. Expressing all exponentials containing $\hat{H}$ in terms of $\hat{H}_0$ and $\mathscr{S}$, we have

$$\begin{aligned} \mathscr{G}(\tau > 0) = -e^{\Omega/T}\,\mathrm{Sp}\,\{&e^{-(\hat{H}_0-\mu\hat{N})/T}\mathscr{S}(1/T)\mathscr{S}^{-1}(\tau_1)e^{(\hat{H}_0-\mu\hat{N})\tau_1}\psi(\mathbf{r}_1) \\ &\times e^{-(\hat{H}_0-\mu\hat{N})\tau_1}\mathscr{S}(\tau_1)\mathscr{S}^{-1}(\tau_2)e^{(\hat{H}_0-\mu\hat{N})\tau_2}\psi^+(\mathbf{r}_2)e^{-(\hat{H}_0-\mu\hat{N})\tau_2}\mathscr{S}(\tau_2)\}, \end{aligned}$$

or, because of (12.1) and (12.5),

$$\mathscr{G}(\tau > 0) = -e^{\Omega/T}\,\mathrm{Sp}\,\{e^{-(\hat{H}_0-\mu\hat{N})/T}\mathscr{S}(1/T, \tau_1)\psi(\mathbf{r}_1, \tau_1)\mathscr{S}(\tau_1, \tau_2)\bar{\psi}(\mathbf{r}_2,\tau_2)\mathscr{S}(\tau_2)\}. \qquad (12.6a)$$

Similarly, for $\tau < 0$ we can write $\mathscr{G}$ in the form

$$\begin{aligned} \mathscr{G}(\tau < 0) = \pm\, e^{\Omega/T}\,\mathrm{Sp}\,\{&e^{-(\hat{H}_0-\mu\hat{N})/T}\mathscr{S}(1/T, \tau_2) \\ &\times\bar{\psi}(\mathbf{r}_2, \tau_2)\mathscr{S}(\tau_2, \tau_1)\psi(\mathbf{r}_1, \tau_1)\mathscr{S}(\tau_1)\}. \end{aligned} \qquad (12.6b)$$

The expressions (12.6a) and (12.6b) can be combined into a single formula

$$\mathscr{G}(\tau) = -e^{\Omega/T}\,\mathrm{Sp}\,\{e^{-(\hat{H}_0-\mu\hat{N})/T}T_\tau(\psi(\mathbf{r}_1, \tau_1)\bar{\psi}(\mathbf{r}_2, \tau_2)\mathscr{S}(1/T))\}, \qquad (12.6c)$$

which follows at once from (12.5) and the definition of the operation of T_τ-ordering.

We must still transform the quantity $e^{\Omega/T}$. To do so, we note that

$$e^{-\Omega/T} = \mathrm{Sp}\,\{e^{-(\hat{H}-\mu\hat{N})/T}\},$$

by definition, which immediately implies

$$e^{-\Omega/T} = \mathrm{Sp}\,\{e^{-(\hat{H}_0-\mu\hat{N})/T}\mathscr{S}(1/T)\}.$$

Thus, we can finally write the expression for $\mathscr{G}$ in the interaction representation in the form

$$\mathscr{G}(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2) = \frac{-\mathrm{Sp}\,\{e^{-(\hat{H}_0-\mu\hat{N})/T}T_\tau(\psi(\mathbf{r}_1, \tau_1)\bar{\psi}(\mathbf{r}_2, \tau_2)\mathscr{S}(1/T))\}}{\mathrm{Sp}\,\{e^{-(\hat{H}_0-\mu\hat{N})/T}\mathscr{S}(1/T)\}},$$

or, introducing the symbol for the Gibbs average over the states of a system of noninteracting particles,

$$\left.\mathscr{G}(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2) = -\frac{\langle T_\tau(\psi(\mathbf{r}_1, \tau_1)\bar{\psi}(\mathbf{r}_2, \tau_2)\mathscr{S})\rangle_0}{\langle\mathscr{S}\rangle_0},\right\} \Longleftarrow \underline{\underline{(12.7)}}$$

Greatly simplifies life.

where

$$\langle\cdots\rangle_0 = \mathrm{Sp}\,\{e^{(\Omega_0+\mu\hat{N}-\hat{H}_0)/T}\cdots\}, \qquad \mathscr{S} \equiv \mathscr{S}(1/T). \qquad (12.8)$$

Step 1 of Dyson's program.

Repeating step by step all the above calculations, we can obtain expressions for the phonon Green's function and for many-particle Green's functions in the interaction representation. For the phonon Green's function, we find

$$\mathscr{D}(1, 2) = -\frac{\langle T_\tau(\varphi(1)\varphi(2)\mathscr{S})\rangle_0}{\langle \mathscr{S}\rangle_0}, \tag{12.9}$$

All Green's functions

and for the two-particle Green's function

$$\mathscr{G}^{\mathrm{II}}(1, 2; 3, 4) = -\frac{\langle T_\tau(\psi(1)\psi(2)\bar{\psi}(3)\bar{\psi}(4)\mathscr{S})\rangle_0}{\langle \mathscr{S}\rangle_0}. \tag{12.10}$$

The formulas for Green's functions depending on a larger number of variables differ from (12.7), (12.9) and (12.10) only by the number of ψ-operators inside the T_τ-product.

Finally, we give the formula relating the thermodynamic potential Ω and the matrix $\mathscr{S}$:

$$\Omega = \Omega_0 - T \ln \langle \mathscr{S}\rangle_0. \tag{12.11}$$

obvious

Here, Ω_0 denotes the potential Ω in the absence of any interaction:

$$\Omega_0 = -T \ln \mathrm{Sp}\,\{e^{-(\mathscr{H}_0 - \mu \hat{N})/T}\}.$$

12.2. Wick's theorem. We now turn to our basic problem, i.e., the calculation of the Green's function for a system of interacting particles. If the interaction between the particles can be regarded as weak, then, using the expression for the temperature Green's function in the interaction representation, we can write the perturbation series with respect to $\tilde{H}_{\mathrm{int}}$ in an exceptionally concise form. The interaction Hamiltonian enters the Green's function only by way of the matrix $\mathscr{S}$. Expanding the exponential in the right-hand side of (12.4) in a power series in $\tilde{H}_{\mathrm{int}}(\tau)$, we obtain

$$\begin{aligned}\mathscr{S} = 1 &- \int_0^{1/T} \tilde{H}_{\mathrm{int}}(\tau')\,d\tau' \\ &+ \frac{1}{2}\int_0^{1/T}\int_0^{1/T} d\tau'\,d\tau''\,T_\tau(\tilde{H}_{\mathrm{int}}(\tau')\tilde{H}_{\mathrm{int}}(\tau'')) - \cdots \qquad (12.12) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\int_0^{1/T}\cdots\int_0^{1/T} d\tau_1\cdots d\tau_n\,T_\tau(\tilde{H}_{\mathrm{int}}(\tau_1)\cdots\tilde{H}_{\mathrm{int}}(\tau_n)).\end{aligned}$$

Perturb. theory begins here

Then, substituting this expansion into the numerator of formula (12.7), we find the perturbation series for the Green's function

Step 2 of Dyson

$$\begin{aligned}\mathscr{G}_{\alpha\beta}(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2) &= -\frac{1}{\langle \mathscr{S}\rangle_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^{1/T}\cdots\int_0^{1/T} d\tau_1'\cdots d\tau_n' \qquad (12.13) \\ &\times \langle T_\tau(\psi_\alpha(\mathbf{r}_1, \tau_1)\bar{\psi}_\beta(\mathbf{r}_2, \tau_2)\tilde{H}_{\mathrm{int}}(\tau_1')\cdots\tilde{H}_{\mathrm{int}}(\tau_n'))\rangle_0.\end{aligned}$$

Of course, the first term of (12.13) is just the free Green's function

$$\mathscr{G}^{(0)} = -\langle T_\tau(\psi(1)\bar{\psi}(2))\rangle_0,$$

calculated in Sec. 11. We shall not expand the matrix $\mathscr{S}$ in the expression $\langle \mathscr{S}\rangle_0$ appearing in the denominator of (12.13), since it will turn out that $\langle \mathscr{S}\rangle_0$ cancels an identical factor in the numerator. Moreover, $\langle \mathscr{S}\rangle_0$ is a constant, independent of $\mathbf{r}$ and τ, and can have no effect on our subsequent considerations.

In all real problems, H_{int} is a product of a certain number (usually not large) of operators $\psi(\mathbf{r}, t)$, $\bar{\psi}(\mathbf{r}, t)$, and perhaps $\varphi(\mathbf{r}, t)$, integrated over the space variables. Therefore, the problem of calculating the Green's function by using perturbation theory reduces to calculating the average value of the T_τ-product of a certain number of ψ-operators, evaluated at various points of space and "time":

$$\langle T_\tau(\psi_\alpha(\mathbf{r}, \tau)\cdots\bar{\psi}_{\alpha'}(\mathbf{r}', \tau')\cdots)\rangle_0. \tag{12.14}$$

We have already encountered a problem of this kind in the preceding chapter, while calculating ordinary Green's functions at the absolute zero of temperature. It was shown there that the average of any number of operators reduces to a sum of products of all possible averages of pairs of operators, where the latter, by definition, equal the free-particle Green's functions (Wick's theorem). As we shall see in a moment, the same situation prevails in the present case.

To convince ourselves of this, we replace the ψ-operators in (12.14) by their Fourier expansions[3] (with respect to $\mathbf{r}$):

$$\begin{aligned}\psi(\mathbf{r}, \tau) &= \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} a_{\mathbf{p}}(\tau)e^{i\mathbf{p}\cdot\mathbf{r} - [\varepsilon_0(\mathbf{p}) - \mu]\tau}, \\ \bar{\psi}(\mathbf{r}, \tau) &= \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} a_{\mathbf{p}}^{+}(\tau)e^{-i\mathbf{p}\cdot\mathbf{r} + [\varepsilon_0(\mathbf{p}) - \mu]\tau}.\end{aligned} \tag{12.15}$$

The operators $a_{\mathbf{p}}(\tau)$ and $a_{\mathbf{p}}^{+}(\tau)$ in (12.15) represent ordinary annihilation and creation operators, and actually do not depend on τ. However, we preserve the argument τ to indicate the places the various operators should occupy in T_τ-products. Substituting the expansions (12.15) into (12.14), we obtain the following expression [except for the exponentials in (12.15)]:

$$\begin{aligned}&\frac{1}{\sqrt{V}}\sum_{\mathbf{p}_1}\frac{1}{\sqrt{V}}\sum_{\mathbf{p}_2}\cdots\frac{1}{\sqrt{V}}\sum_{\mathbf{p}_1'}\frac{1}{\sqrt{V}}\sum_{\mathbf{p}_2'}\cdots \qquad (12.16) \\ &\times \langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a_{\mathbf{p}_2}(\tau_2)\cdots a_{\mathbf{p}_1'}^{+}(\tau_1')a_{\mathbf{p}_2'}^{+}(\tau_2')\cdots)\rangle_0.\end{aligned}$$

In the sum over $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_1', \mathbf{p}_2', \cdots$, the only nonzero terms are those containing equal numbers of annihilation and creation operators referring to the same momenta. In particular, the terms containing only one annihilation operator and one creation operator with the same momenta are nonzero, as are the terms

$$\frac{1}{V}\sum_{\substack{\mathbf{p}_1 \\ \mathbf{p}_1 \neq \mathbf{p}_2 \neq \cdots}}\frac{1}{V}\sum_{\mathbf{p}_2}\cdots\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a_{\mathbf{p}_2}(\tau_2)\cdots a_{\mathbf{p}_1}^{+}(\tau_1')a_{\mathbf{p}_2}^{+}(\tau_2)\cdots)\rangle_0, \tag{12.17a}$$

---

[3] The validity of (12.15) can be verified most easily by using the definition of the operators in the interaction representation and the identities (11.10).

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and other terms differing from (12.17a) by a permutation of the momenta $\mathbf{p}_1, \mathbf{p}_2, \cdots$ indexing the a^+-operators. When there are several annihilation and creation operators with the same momenta (say two), the corresponding nonzero terms in the sum have the form

$$\frac{1}{V}\cdot\frac{1}{V}\sum_{\substack{\mathbf{p}_1\\ \mathbf{p}_1\neq\mathbf{p}_2\neq\cdots}}\frac{1}{V}\sum_{\mathbf{p}_2}\cdots\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a_{\mathbf{p}_1}(\tau_2)a_{\mathbf{p}_2}(\tau_3)\cdots a^+_{\mathbf{p}_1}(\tau_1')a^+_{\mathbf{p}_1}(\tau_2')a^+_{\mathbf{p}_2}(\tau_3')\cdots)\rangle_0. \tag{12.17b}$$

The expressions (12.17a) have a special feature which distinguishes them from all the others, i.e., the number of factors $1/V$ in (12.17a) is the same as the number of summations, whereas in all the other expressions this number of factors is larger. Suppose, after having carried out the average $\langle\cdots\rangle_0$, we let the volume V of our system approach infinity, while keeping the particle density N/V constant.[4] In the limit as $V\to\infty$, the sum (12.17a) remains finite, and in fact becomes an expression involving integrals (with respect to the momenta) of various combinations of Fermi or Bose functions.[5] On the other hand, in expressions of the form (12.17b), besides these integrals with respect to the momenta there remains a certain number of extra factors $1/V$, and hence these expressions vanish as $V\to\infty$. Thus, the only terms in the sum (12.16) which remain in the limit as $V\to\infty$ are the terms of the form (12.17a), where all the annihilation and creation operators are indexed by different momenta. This means that in calculating the expression

$$\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a_{\mathbf{p}_2}(\tau_2)\cdots a^+_{\mathbf{p}_1'}(\tau_1')a^+_{\mathbf{p}_2'}(\tau_2')\cdots)\rangle_0,$$

we can actually average each pair of operators $a_\mathbf{p}$ and $a_\mathbf{p}^+$ separately. As a result, the average value of a T_τ-product of a large number of operators can be expressed as a sum of all possible averages of pairs of operators. For example,

$$\begin{aligned}\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a_{\mathbf{p}_2}(\tau_2)a^+_{\mathbf{p}_1'}(\tau_1')a^+_{\mathbf{p}_2'}(\tau_2'))\rangle_0 \\ = \langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a^+_{\mathbf{p}_2'}(\tau_2'))\rangle_0\langle T_\tau(a_{\mathbf{p}_2}(\tau_2)a^+_{\mathbf{p}_1'}(\tau_1'))\rangle_0 \qquad (12.18a)\\ \mp \langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a^+_{\mathbf{p}_1'}(\tau_1'))\rangle_0\langle T_\tau(a_{\mathbf{p}_2}(\tau_2)a^+_{\mathbf{p}_2'}(\tau_2'))\rangle_0,\end{aligned}$$

where the minus sign corresponds to Fermi statistics, and the plus sign to Bose statistics.[6]

In the coordinate representation, these results mean that the average of a T_τ-product of a certain number of ψ-operators decomposes into a sum of products of all possible averages involving pairs of operators $\psi, \bar\psi$. For example, instead of (12.18a), we have

$$\begin{aligned}&\langle T_\tau(\psi(\mathbf{r}_1,\tau_1)\psi(\mathbf{r}_2,\tau_2)\bar\psi(\mathbf{r}_1',\tau_1')\bar\psi(\mathbf{r}_2',\tau_2'))\rangle_0\\
&\quad=\frac{1}{V^2}\sum_{\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_1,\mathbf{p}_2} e^{i(\mathbf{p}_1\cdot\mathbf{r}_1+\mathbf{p}_2\cdot\mathbf{r}_2-\mathbf{p}_1'\cdot\mathbf{r}_1'-\mathbf{p}_2'\cdot\mathbf{r}_2')}e^{-[\varepsilon_0(\mathbf{p}_1)-\mu]\tau_1-[\varepsilon_0(\mathbf{p}_2)-\mu]\tau_2}\\
&\qquad\times e^{[\varepsilon_0(\mathbf{p}_1')-\mu]\tau_1'+[\varepsilon_0(\mathbf{p}_2')-\mu]\tau_2'}\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a_{\mathbf{p}_2}(\tau_2)a^+_{\mathbf{p}_1'}(\tau_1')a^+_{\mathbf{p}_2'}(\tau_2'))\rangle_0\\
&\quad=\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2'} e^{i(\mathbf{p}_1\cdot\mathbf{r}_1-\mathbf{p}_2'\cdot\mathbf{r}_2')-[\varepsilon_0(\mathbf{p}_1)-\mu]\tau_1+[\varepsilon_0(\mathbf{p}_2')-\mu]\tau_2'}\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a^+_{\mathbf{p}_2'}(\tau_2'))\rangle_0\\
&\qquad\times\frac{1}{V}\sum_{\mathbf{p}_2,\mathbf{p}_1'} e^{i(\mathbf{p}_2\cdot\mathbf{r}_2-\mathbf{p}_1'\cdot\mathbf{r}_1')-[\varepsilon_0(\mathbf{p}_2)-\mu]\tau_2+[\varepsilon_0(\mathbf{p}_1')-\mu]\tau_1'}\langle T_\tau(a_{\mathbf{p}_2}(\tau_2)a^+_{\mathbf{p}_1'}(\tau_1'))\rangle_0\\
&\qquad\mp\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2'} e^{i(\mathbf{p}_1\cdot\mathbf{r}_1-\mathbf{p}_1'\cdot\mathbf{r}_1')-[\varepsilon_0(\mathbf{p}_1)-\mu]\tau_1+[\varepsilon_0(\mathbf{p}_1')-\mu]\tau_1'}\langle T_\tau(a_{\mathbf{p}_1}(\tau_1)a^+_{\mathbf{p}_1'}(\tau_1'))\rangle_0\\
&\qquad\times\frac{1}{V}\sum_{\mathbf{p}_2,\mathbf{p}_2'} e^{i(\mathbf{p}_2\cdot\mathbf{r}_2-\mathbf{p}_2'\cdot\mathbf{r}_2')-[\varepsilon_0(\mathbf{p}_2)-\mu]\tau_2+[\varepsilon_0(\mathbf{p}_2')-\mu]\tau_2'}\langle T_\tau(a_{\mathbf{p}_2}(\tau_2)a^+_{\mathbf{p}_2'}(\tau_2'))\rangle_0\\
&\quad=\langle T_\tau(\psi(\mathbf{r}_1,\tau_1)\bar\psi(\mathbf{r}_2',\tau_2'))\rangle_0\langle T_\tau(\psi(\mathbf{r}_2,\tau_2)\bar\psi(\mathbf{r}_1',\tau_1'))\rangle_0\\
&\qquad\mp\langle T_\tau(\psi(\mathbf{r}_1,\tau_1)\bar\psi(\mathbf{r}_1',\tau_1'))\rangle_0\langle T_\tau(\psi(\mathbf{r}_2,\tau_2)\bar\psi(\mathbf{r}_2',\tau_2'))\rangle_0,\end{aligned}\tag{12.18b}$$

and similar relations hold for a larger number of operators.

The averages appearing in the right-hand side of (12.18b) are just free-particle temperature Green's functions (except possibly for sign). Thus, in calculating temperature Green's functions, we encounter the same situation as found in the case $T = 0$. In fact, the Green's function $\mathscr{G}$ satisfies the expansion (12.13), which, except for the factors of i^n and the limits of τ-integration, is identical with the expansion (8.9) for the function G. Moreover, to calculate the averages $\langle T_\tau(\cdots)\rangle_0$ appearing in (12.13), we can use Wick's theorem, just as before, which allows us to express these averages in terms of averages of pairs of annihilation and creation operators. It should be noted that in the present technique, there is no concept of a normal product, and Wick's theorem does not hold for T_τ-products themselves, but only for average values of these products.

Using Wick's theorem to write down any term of the series (12.13), and replacing $\langle T_\tau(\psi_1\bar\psi)\rangle_0$ by the free Green's function

$$\mathscr{G}^{(0)}_{\alpha\beta}(\mathbf{r}_1-\mathbf{r}_2,\tau_1-\tau_2) = -\langle T_\tau(\psi_\alpha(\mathbf{r}_1,\tau_1)\bar\psi_\beta(\mathbf{r}_2,\tau_2))\rangle_0,$$

we arrive at expressions which have exactly the same structure as the corresponding series for $T = 0$. This allows us to describe the various approximations of the perturbation series by using the same Feynman diagrams as used in the preceding chapter, and the only change is in the rules by which analytical expressions are associated with elements of the diagrams. In the

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[4] When this is done, the sums are replaced by integrals, according to the rule

$$\frac{1}{V}\sum\cdots\to\frac{1}{(2\pi)^3}\int\cdots$$

[5] We have already encountered the simplest example of this kind in Sec. 11, in calculating the free Green's function $\mathscr{G}^{(0)} = -\langle T_\tau(\psi(1)\bar\psi(2))\rangle_0$.

[6] It can be shown that this rule also holds for the average of a T_τ-product of several $a_\mathbf{p}$ and $a_\mathbf{p}^+$ with the same momenta [as in (12.17b)], excluding the case of the operators a_0 and a_0^+ for a Bose system below the condensation temperature (see p. 110). For simplicity, the proof will not be given here.

present case, with each line of a diagram we have to associate a free-particle temperature Green's function $\mathscr{G}^{(0)}$, instead of a function $G^{(0)}$. Moreover, integration with respect to the time t (from $-\infty$ to ∞) at every vertex of a diagram has to be replaced by integration with respect to the "time" τ (from 0 to $1/T$).

So far, we have tacitly assumed that as the volume of the system goes to infinity (with the density kept constant), all the free-particle Green's functions and the integrals involving them remain finite. In particular, this justifies our neglect of terms of the form (12.17b) in the limit as $V \to \infty$. The situation changes drastically in the case of Bose systems below the condensation temperature T_c or Fermi systems exhibiting superconductivity. In the case of a Bose gas with $T < T_c$, the annihilation and creation operators for particles in the state of zero momentum are proportional to the square root of the volume:

$$a_0 \sim a_0^+ \propto \sqrt{N} \propto \sqrt{V}.$$

Then the terms of the type (12.17b), with several a_0 and a_0^+, remain finite as $V \to \infty$, and moreover do not obey Wick's theorem (cf. footnote 6, p. 108). A similar situation occurs for superconductors. In both cases, we have to use a special technique, which will be described in separate chapters (Chaps. 5 and 7).

We now return to the case where the usual diagram technique is applicable. Just as in the preceding chapter, the diagrams for the Green's functions have two external lines. One of these external lines begins at the point $\mathbf{r}_1, \tau_1$ corresponding to the coordinates of the operator $\psi_\alpha(\mathbf{r}_1, \tau_1)$, while the other external line terminates at the point $\mathbf{r}_2, \tau_2$ corresponding to the operator $\bar{\psi}_\beta(\mathbf{r}_2, \tau_2)$. As before, the diagrams for the function $\mathscr{G}$ can be divided into two groups, i.e., connected diagrams and disconnected diagrams. Using an argument exactly analogous to that given for the case $T = 0$, we can verify that the contribution from the disconnected diagrams cancels the denominator in formula (12.7). As a result, we find that

Step#3

$$\boxed{\mathscr{G}_{\alpha\beta}(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2) = -\langle T_\tau(\psi_\alpha(\mathbf{r}_1, \tau_1)\bar{\psi}_\beta(\mathbf{r}_2, \tau_2)\mathscr{A})\rangle_{\text{con}},} \quad (12.19)$$

where $\langle \cdots \rangle_{\text{con}}$ denotes the contribution from connected diagrams only. A similar result holds for many-particle Green's functions, since the derivation nowhere uses the fact that the diagrams have two external lines. In other words, $\langle \mathscr{A} \rangle_0$ can be omitted in the denominators of the corresponding formulas [of the type (12.10)], and we need only take account of connected diagrams when calculating averages.

Just as in the preceding chapter, each diagram appears in the series for $\mathscr{G}$ with some coefficient of the form λ^m, which does not depend in an essential way on the order m of the diagram (see p. 71). This fact is very important when summing infinite sequences of diagrams.

13. The Diagram Technique in Coordinate Space. Examples

The basic result of the preceding section is our proof of the fact that in calculating temperature Green's functions, we can apply the usual technique of Feynman diagrams. The basic elements of every such diagram are the lines representing the Green's functions of free particles or of phonons. Just as in Chap. 2, we represent the Green's function of a particle by a solid line (see Fig. 32). The direction of the line is indicated by an arrow, i.e., the line begins at the point with coordinates $\mathbf{r}_1, \tau_1$ and spin projection α_1 (the point corresponding to the operator ψ in the definition of the function $\mathscr{G}$), and terminates at the point $\mathbf{r}_2, \tau_2, \alpha_2$ (corresponding to the operator $\bar{\psi}$). In arguments of Green's functions, the coordinates of initial points are written on the left and coordinates of end points are written on the right. Thus, the line in Fig. 32(a) represents the Green's function

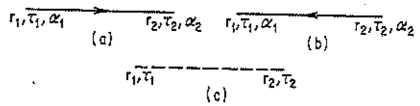

FIGURE 32

$$\mathscr{G}^{(0)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2) \equiv \mathscr{G}^{(0)}_{\alpha_1\alpha_2}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2),$$

while the line in Fig. 32(b) represents the Green's function

$$\mathscr{G}^{(0)}_{\alpha_2\alpha_1}(\mathbf{r}_2, \tau_2; \mathbf{r}_1, \tau_1) \equiv \mathscr{G}^{(0)}_{\alpha_2\alpha_1}(\mathbf{r}_2 - \mathbf{r}_1, \tau_2 - \tau_1).$$

We use a dashed line to represent the Green's function of a phonon, as in Fig. 32(c). The direction of a phonon line need not be indicated, since as we have seen in Sec. 11, $\mathscr{D}^{(0)}$ is an even function of $\mathbf{r}_1 - \mathbf{r}_2$ and $\tau_1 - \tau_2$.

As before, we integrate over the coordinates of the "vertices," i.e., the points where the lines intersect. The integration with respect to $\mathbf{r}$ ranges over all space, and the integration with respect to τ is between the limits 0 and $1/T$. At the vertices, we also sum over the spin variables.

The specific form of the diagrams depends on the nature of the interaction between the particles. To construct diagrams, we have to use Wick's theorem, according to which averages of T_τ-products of operators appearing in the perturbation series (12.13) for the Green's function can be represented as a sum of products of averages involving pairs of operators. These latter averages are related to the free-particle Green's functions by the formulas

$$\begin{aligned} \langle T_\tau(\psi_\alpha(\mathbf{r}_1, \tau_1)\bar{\psi}_\beta(\mathbf{r}_2, \tau_2))\rangle &= -\mathscr{G}^{(0)}_{\alpha\beta}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2), \\ \langle T_\tau(\bar{\psi}_\beta(\mathbf{r}_2, \tau_2)\psi_\alpha(\mathbf{r}_1, \tau_1))\rangle &= \pm\mathscr{G}^{(0)}_{\alpha\beta}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2), \end{aligned} \quad (13.1)$$

where the plus sign corresponds to fermions, and the minus sign to bosons.

Similarly, the average of a product of two phonon operators can be expressed in terms of the function $\mathscr{D}^{(0)}$:

$$\langle T_\tau(\varphi(\mathbf{r}_1, \tau_1)\varphi(\mathbf{r}_2, \tau_2))\rangle = -\mathscr{D}^{(0)}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2). \qquad (13.2)$$

We now consider different kinds of interactions.

A. Two-particle interactions. Suppose the particles of the system are acted upon by two-particle forces described by a potential $U(\mathbf{r}_1 - \mathbf{r}_2)$. Then the Hamiltonian has the form

$$\hat{H}_{\text{int}}(\tau) = \frac{1}{2}\int d\mathbf{r}_1\, d\mathbf{r}_2\, \bar{\psi}_\alpha(\mathbf{r}_1, \tau)\bar{\psi}_\beta(\mathbf{r}_2, \tau)U(\mathbf{r}_1 - \mathbf{r}_2)\psi_\beta(\mathbf{r}_2, \tau)\psi_\alpha(\mathbf{r}_1, \tau) \qquad (13.3)$$

in the interaction representation. Instead of the potential $U(\mathbf{r}_1 - \mathbf{r}_2)$, it is convenient to introduce a potential $\mathscr{V}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2)$, depending on the "time" τ and defined by the formula

$$\mathscr{V}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2) = U(\mathbf{r}_1 - \mathbf{r}_2)\delta(\tau_1 - \tau_2). \qquad (13.4)$$

Using (13.4), we can write the expression (12.4) for the matrix $\mathscr{A}$ in the symmetric form

$$\mathscr{A} = T_\tau \exp\left\{-\frac{1}{2}\int d\mathbf{r}_1\, d\mathbf{r}_2\, d\tau_1\, d\tau_2\, \bar{\psi}_\alpha(\mathbf{r}_1, \tau_1)\bar{\psi}_\beta(\mathbf{r}_2, \tau_2) \times \mathscr{V}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2)\psi_\beta(\mathbf{r}_2, \tau_2)\psi_\alpha(\mathbf{r}_1, \tau_1)\right\}.$$

Next, we calculate the correction to the Green's function which is of the first order in U, obtaining[7]

$$\mathscr{G}^{(1)}_{\alpha\beta}(x - y) = \frac{1}{2}\int d^4z_1\, d^4z_2 \times \langle T_\tau(\psi_\alpha(x)\bar{\psi}_\beta(y)\mathscr{V}(z_1 - z_2)\bar{\psi}_{\gamma_1}(z_1)\bar{\psi}_{\gamma_2}(z_2)\psi_{\gamma_2}(z_2)\psi_{\gamma_1}(z_1))\rangle. \qquad (13.5)$$

According to Wick's theorem, the average $\langle \cdots \rangle$ in the right-hand side of (13.5) can be written as a sum of four terms

$$\langle T_\tau(\psi_\alpha(x)\bar{\psi}_\beta(y))\rangle\langle\bar{\psi}_{\gamma_1}(z_1)\psi_{\gamma_1}(z_1)\rangle\langle\bar{\psi}_{\gamma_2}(z_2)\psi_{\gamma_2}(z_2)\rangle, \qquad \text{(a)}$$

$$\mp \langle T_\tau(\psi_\alpha(x)\bar{\psi}_\beta(y))\rangle\langle\bar{\psi}_{\gamma_2}(z_2)\psi_{\gamma_1}(z_1)\rangle\langle\bar{\psi}_{\gamma_1}(z_1)\psi_{\gamma_2}(z_2)\rangle, \qquad \text{(b)}$$

$$\langle T_\tau(\psi_\alpha(x)\bar{\psi}_{\gamma_1}(z_1))\rangle\langle T_\tau(\psi_{\gamma_1}(z_1)\bar{\psi}_\beta(y))\rangle\langle\bar{\psi}_{\gamma_2}(z_2)\psi_{\gamma_2}(z_2)\rangle, \qquad \text{(c)}$$

$$\mp \langle T_\tau(\psi_\alpha(x)\bar{\psi}_{\gamma_1}(z_1))\rangle\langle\bar{\psi}_{\gamma_2}(z_2)\psi_{\gamma_1}(z_1)\rangle\langle T_\tau(\psi_{\gamma_2}(z_2)\bar{\psi}_\beta(y))\rangle, \qquad \text{(d)}$$

and of four more terms, obtained from (a)–(d) by making the substitution $z_1 \to z_2, \gamma_1 \to \gamma_2$. The contribution from these last four terms to the integral in (13.5) is obviously the same as the contribution from (a)–(d), and this simply leads to the disappearance of the factor $\frac{1}{2}$ in front of the integral.

Using (13.1) to replace the averages $\langle T_\tau(\cdots)\rangle$ by suitable Green's functions $\mathscr{G}^{(0)}$, we find that the first-order correction is a sum of the following four terms:

$$-\mathscr{G}^{(0)}_{\alpha\beta}(x - y)\int d^4z_1\, d^4z_2\, \mathscr{G}^{(0)}_{\gamma_1\gamma_1}(0)\mathscr{G}^{(0)}_{\gamma_2\gamma_2}(0)\mathscr{V}(z_1 - z_2), \qquad \text{(a)}$$

$$\pm\mathscr{G}^{(0)}_{\alpha\beta}(x - y)\int d^4z_1\, d^4z_2\, \mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_1}(z_2 - z_1)\mathscr{V}(z_1 - z_2), \qquad \text{(b)}$$

$$\pm\int d^4z_1\, d^4z_2\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\beta}(z_1 - y)\mathscr{G}^{(0)}_{\gamma_2\gamma_2}(0)\mathscr{V}(z_1 - z_2), \qquad \text{(c)}$$

$$-\int d^4z_1\, d^4z_2\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\beta}(z_2 - y)\mathscr{V}(z_1 - z_2). \qquad \text{(d)}$$

It should be noted that the quantity $\mathscr{G}^{(0)}(\mathbf{r}_1 - \mathbf{r}_2, 0)$ is always evaluated as $\lim_{\tau \to +0} \mathscr{G}^{(0)}(\mathbf{r}_1 - \mathbf{r}_2, -\tau)$.

In constructing diagrams, we represent $\mathscr{V}(z_1 - z_2)$ by a wavy line. Then the expressions (a)–(d) correspond to the diagrams shown in Fig. 33. Diagrams (a) and (b) are disconnected, and as shown in the preceding section, they should not be taken into account when calculating Green's functions. Thus, the only contribution to the first-order correction is made by diagrams (c) and (d), and by the "topologically equivalent" diagrams differing from them by permutation of the vertex coordinates z_1 and z_2 (recall that all topologically equivalent diagrams make the same contribution). It should be pointed out that in the case of Fermi statistics, the expressions corresponding to diagrams (c) and (d) have opposite signs. This fact is associated with the presence of a closed loop in diagram (c). For a diagram of arbitrary order, it can be shown that any closed fermion loop (not necessarily formed of a single line, as in the present case) leads to a factor of -1 in the corresponding analytical expression.

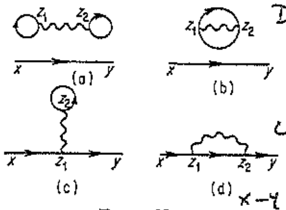

FIGURE 33

Disconne

Connected

x – y External point.

z1 z2 internal

We now state the general rules which are used to calculate the correction of order n:

1. Form all connected, topologically nonequivalent diagrams with $2n$ vertices and two external lines, where two solid lines and one wavy line meet at each vertex.
2. With each solid line associate a Green's function $\mathscr{G}^{(0)}_{\alpha\beta}(x - y)$, where x, α are the coordinates of the initial point of the line, and y, β are the coordinates of its end point.
3. With each wavy line associate a generalized potential $\mathscr{V}(x - y)$.

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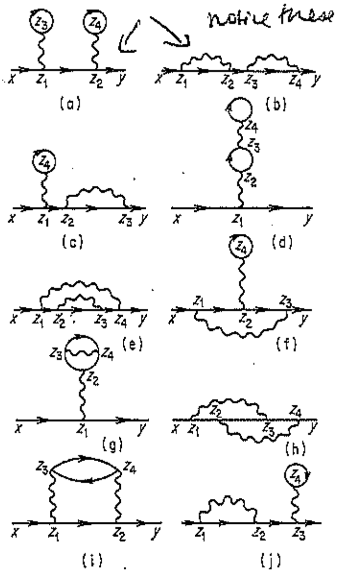
[7] In the rest of this section, we use lightface Latin letters to denote the set of four variables, $x = (\mathbf{r}, \tau)$. Thus, $\mathscr{G}(x - y) = \mathscr{G}(\mathbf{x} - \mathbf{y}, \tau_1 - \tau_2)$ and $d^4x = d\mathbf{r}\, d\tau$.

4. Integrate over all the vertex coordinates z ($d^4z = d\mathbf{r}\,d\tau$), and sum over all internal spin variables α.
5. Multiply the resulting expression by $(-1)^{n+F}$, where F is the number of closed fermion loops.
6. If there are any Green's functions $\mathscr{G}^{(0)}(0)$ whose "time" arguments are the same, interpret them as $\lim_{\tau \to +0} \mathscr{G}^{(0)}(\mathbf{r}_1 - \mathbf{r}_2, -\tau)$.



notice these repeat lower order motifs. Can be absorbed into proper self energy

FIGURE 34

For example, consider the second-order correction. All possible connected, topologically nonequivalent diagrams with four vertices are shown in Fig. 34. Using the above rules, we can easily write down the following analytical expressions corresponding to these diagrams:

$$\int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\beta}(z_2 - y) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_3}(0)\mathscr{G}^{(0)}_{\gamma_4\gamma_4}(0)\mathscr{V}(z_1 - z_3)\mathscr{V}(z_2 - z_4), \tag{a}$$

$$\int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_4}(z_3 - z_4)\mathscr{G}^{(0)}_{\gamma_4\beta}(z_4 - y)\mathscr{V}(z_1 - z_2)\mathscr{V}(z_3 - z_4), \tag{b}$$

$$\mp \int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\beta}(z_3 - y)\mathscr{G}^{(0)}_{\gamma_4\gamma_4}(0)\mathscr{V}(z_1 - z_4)\mathscr{V}(z_2 - z_3), \tag{c}$$

$$\int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\beta}(z_1 - y)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_2}(z_3 - z_2)\mathscr{G}^{(0)}_{\gamma_4\gamma_4}(0)\mathscr{V}(z_1 - z_2)\mathscr{V}(z_3 - z_4), \tag{d}$$

$$\int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_4}(z_3 - z_4)\mathscr{G}^{(0)}_{\gamma_4\beta}(z_4 - y)\mathscr{V}(z_1 - z_4)\mathscr{V}(z_2 - z_3), \tag{e}$$

$$\mp \int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\beta}(z_3 - y)\mathscr{G}^{(0)}_{\gamma_4\gamma_4}(0)\mathscr{V}(z_1 - z_3)\mathscr{V}(z_2 - z_4), \tag{f}$$

$$\mp \int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\beta}(z_1 - y)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_4}(z_3 - z_4)\mathscr{G}^{(0)}_{\gamma_4\gamma_2}(z_4 - z_2)\mathscr{V}(z_1 - z_2)\mathscr{V}(z_3 - z_4), \tag{g}$$

$$\int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_4}(z_3 - z_4)\mathscr{G}^{(0)}_{\gamma_4\beta}(z_4 - y)\mathscr{V}(z_1 - z_3)\mathscr{V}(z_2 - z_4), \tag{h}$$

$$\mp \int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\beta}(z_2 - y) \times \mathscr{G}^{(0)}_{\gamma_3\gamma_4}(z_3 - z_4)\mathscr{G}^{(0)}_{\gamma_4\gamma_3}(z_4 - z_3)\mathscr{V}(z_1 - z_3)\mathscr{V}(z_2 - z_4), \tag{i}$$

$$\mp \int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\gamma_3}(z_2 - z_3) \times \mathscr{G}_{\gamma_3\beta}(z_3 - y)\mathscr{G}_{\gamma_4\gamma_4}(0)\mathscr{V}(z_1 - z_2)\mathscr{V}(z_3 - z_4). \tag{j}$$

In the case of two-particle interactions, perturbation theory can be cast in another, more symmetric form, which is particularly convenient when the interaction forces depend not only on the distances between the particles but also on their spins. The Hamiltonian for such an interaction has the form

$$\bar{H}_{\text{int}}(\tau) = \frac{1}{2}\int d\mathbf{r}_1\, d\mathbf{r}_2\, \bar{\psi}_\alpha(\mathbf{r}_1, \tau)\bar{\psi}_\beta(\mathbf{r}_2, \tau)U_{\alpha\beta,\delta\gamma}(\mathbf{r}_1 - \mathbf{r}_2)\psi_\gamma(\mathbf{r}_2, \tau)\psi_\delta(\mathbf{r}_1, \tau). \tag{13.6}$$

We write the integral

$$\int_0^{1/T} H_{\text{int}}(\tau)\, d\tau$$

appearing in the expression for $\mathscr{A}$ as

$$\frac{1}{4}\int_0^{1/T} d\tau_1 \cdots \int_0^{1/T} d\tau_4 \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4 \times \bar{\psi}_{\gamma_1}(\mathbf{r}_1, \tau_1)\bar{\psi}_{\gamma_2}(\mathbf{r}_2, \tau_2)\mathscr{T}^{(0)}_{\gamma_1\gamma_2,\gamma_3\gamma_4}(\mathbf{r}_1, \tau_1, \mathbf{r}_2, \tau_2; \mathbf{r}_3, \tau_3, \mathbf{r}_4, \tau_4)\psi_{\gamma_4}(\mathbf{r}_4, \tau_4)\psi_{\gamma_3}(\mathbf{r}_3, \tau_3),$$

which is symmetric in all four variables, or as

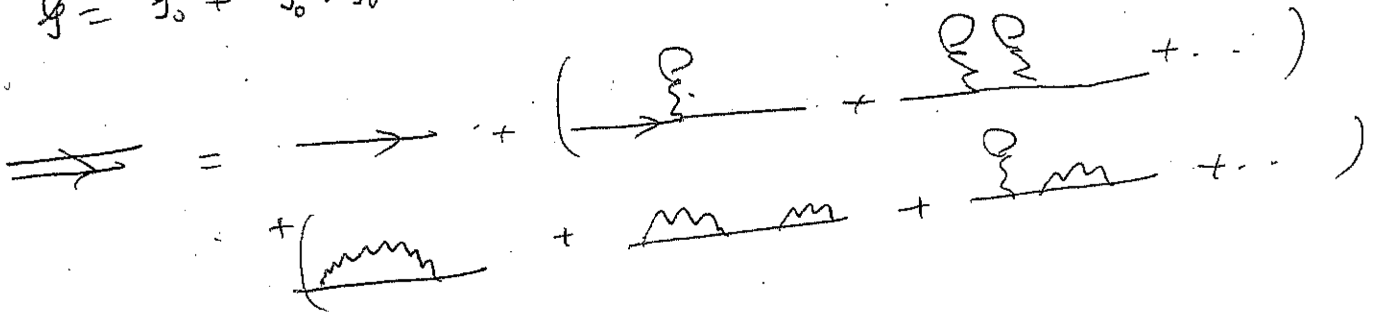
$$\frac{1}{4}\int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \bar{\psi}_{\gamma_1}(z_1)\bar{\psi}_{\gamma_2}(z_2)\mathscr{T}^{(0)}_{\gamma_1\gamma_2,\gamma_3\gamma_4}(z_1, z_2; z_3, z_4)\psi_{\gamma_4}(z_4)\psi_{\gamma_3}(z_3), \tag{13.7}$$

Comment

after page 125

$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 V \mathcal{G}_0 + \ldots$

$G_0 \Sigma_{\text{reducible}} G_0$



Schematically $G = G_0 + G_0 \Sigma_{\text{Reducible}} G_0$

$= G_0 + G_0 \Sigma_i G$ (by expansion this leads to, provided Σ_i has no repeats, i.e. is irreducible)

Think Geometric series $G^{-1} = G_0^{-1} - \Sigma_i$ $G = \frac{G_0}{1-\Sigma_i G_0} = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \ldots$

2. With each solid line associate a free-particle Green's function $\mathscr{G}^{(0)}_{\alpha\beta}(x-y)$, and with each dashed line associate a function $\mathscr{D}^{(0)}(x-y)$.
3. Integrate over the coordinates of all vertices (with respect to both $\mathbf{r}$ and τ), and sum over the corresponding spin variables.

FIGURE 41

4. Multiply the resulting expression by $g^{2n}(-1)^{n+F}$, where F is the number of closed fermion loops.

For example, the fourth-order correction to the phonon Green's function corresponding to the diagram in Fig. 41 is

$$\mp g^4 \int d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, \mathscr{D}^{(0)}(x-z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1-z_2)\mathscr{G}^{(0)}_{\gamma_3\gamma_1}(z_3-z_1)$$
$$\times \mathscr{D}^{(0)}(z_2-z_3)\mathscr{G}^{(0)}_{\gamma_2\gamma_4}(z_2-z_4)\mathscr{G}^{(0)}_{\gamma_4\gamma_3}(z_4-z_3)\mathscr{D}^{(0)}(z_4-y).$$

14. The Diagram Technique in Momentum Space

14.1. Transformation to momentum space. The diagram technique in coordinate space developed in the preceding section turns out to be quite unsuitable for making explicit calculations. It will be recalled that the success of field-theory methods at the absolute zero of temperature is chiefly due to the highly automatic way in which calculations can be performed. For $T = 0$, this was achieved by expanding all quantities appearing in the theory in four-dimensional Fourier integrals. However, in Matsubara's technique (described above), this automatic way of doing things is no longer possible, since the variable τ varies over a finite interval from 0 to $1/T$, so that a transformation to a Fourier-integral representation is impossible. The application of Matsubara's technique in the coordinate representation is very difficult because of the fact that $\mathscr{G}^{(0)}$ and $\mathscr{D}^{(0)}$ are discontinuous functions of the variable τ, and hence all integrals with respect to τ involve a large number of regions of integration, whose number increases very rapidly with n, the order of the approximation.

We now show that Matsubara's technique can be greatly simplified by expanding all quantities depending on τ in *Fourier series* with respect to τ [see Abrikosov, Gorkov and Dzyaloshinski (A5), Fradkin (F2)]. The temperature Green's function $\mathscr{G}$ (or $\mathscr{D}$) is a function of the difference variable $\tau_1 - \tau_2$, and as such is defined in the interval $[-1/T, 1/T]$. Expanding $\mathscr{G}(\tau)$ in Fourier series, we obtain

$$\left.\begin{aligned} \mathscr{G}(\tau) &= T\sum_n e^{-i\omega_n\tau}\mathscr{G}(\omega_n), \\ \mathscr{G}(\omega_n) &= \frac{1}{2}\int_{-1/T}^{1/T} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau, \qquad \omega_n = n\pi T. \end{aligned}\right\} \tag{14.1}$$

Our problem involves going over to a Fourier representation in the expressions (given in Sec. 13) for the corrections to the Green's functions. It is most desirable that the methods used to do this should not introduce any extra complications into the formulas, e.g., that no extra factors depending on the "frequencies" ω_n should appear.

As we now show, this way of dealing with the case $T \neq 0$ leads to only a slight modification of the situation for the case $T = 0$. As proved in Sec. 11, the Green's functions have the general property that their values for $\tau < 0$ are related by simple formulas to their values for $\tau > 0$ [see (11.8) and (11.8′)]. These formulas imply that the Fourier components $\mathscr{G}(\omega_n)$ [or $\mathscr{D}(\omega_n)$] of the Green's functions for bosons and phonons are nonzero only for "even" frequencies $\omega_n = 2n\pi T$, whereas for fermions $\mathscr{G}(\omega_n)$ is nonzero only for "odd" frequencies $\omega_n = (2n+1)\pi T$. To see this, we first note that

$$\begin{aligned} \mathscr{G}(\omega_n) &= \frac{1}{2}\int_{-1/T}^{1/T} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau \\ &= \frac{1}{2}\int_0^{1/T} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau + \frac{1}{2}\int_{-1/T}^{0} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau. \end{aligned}$$

Then, substituting (11.8) for $\mathscr{G}(\tau < 0)$ in the second integral, and making the change of variables $\tau' = \tau + (1/T)$, we obtain

$$\begin{aligned} \mathscr{G}(\omega_n) &= \frac{1}{2}\int_0^{1/T} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau \mp \frac{1}{2}\int_{1/T}^{0} e^{i\omega_n\tau}\mathscr{G}\left(\tau + \frac{1}{T}\right) \\ &= \frac{1}{2}(1 \mp e^{i\omega_n/T})\int_0^{1/T} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau, \end{aligned}$$

which implies the assertion just made. In fact, we always have

$$\mathscr{G}(\omega_n) = \int_0^{1/T} e^{i\omega_n\tau}\mathscr{G}(\tau)\,d\tau, \tag{14.2}$$

where

$$\omega_n = \begin{cases} (2n+1)\pi T & \text{for fermions,} \\ 2n\pi T & \text{for bosons.} \end{cases}$$

← already checked in class

Next, we substitute the Fourier-series expansion (14.1) into all appropriate terms of the perturbation series, simultaneously taking Fourier transforms with respect to the space variables. In other words, we write

$$\begin{aligned} \mathscr{G}(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\cdot\mathbf{r}}\mathscr{G}(\mathbf{p})\,d\mathbf{p}, \\ \mathscr{G}(\mathbf{p}) &= \int e^{-i\mathbf{p}\cdot\mathbf{r}}\mathscr{G}(\mathbf{r})\,d\mathbf{r}, \end{aligned} \tag{14.3}$$

so that as far as the space variables are concerned, the transformation is carried out in exactly the same way as for $T = 0$. We note that an even number of fermion lines meet at every point whose coordinates are involved in an integration. It follows that in evaluating the integral

$$\int_0^{1/T} d\tau\, e^{i\tau\Sigma\omega_n}, \tag{14.4}$$

at each vertex, the sum of "frequencies" $\Sigma\,\omega_n$ in the exponent is always "even," i.e., $\Sigma\,\omega_n = 2N\pi T$, where N is an integer. In this case, the integral (14.4) equals

$$\frac{1}{T}\delta_{\Sigma\omega_n}, \qquad \delta_{\omega_n} = \begin{cases} 1 & \text{for} \quad \omega_n = 0, \\ 0 & \text{for} \quad \omega_n \neq 0. \end{cases} \qquad (14.5)$$

Thus, the situation here is essentially the same as for $T = 0$. It will be recalled that for $T = 0$, integration over the space-time coordinates of the vertices gives rise to δ-functions of frequency and momentum, expressing conservation of the energy and momentum of certain virtual processes. For $T \neq 0$, each δ-function of the frequency is replaced by a Kronecker delta δ_{ω_n}, expressing conservation of the discrete "frequency" ω_n. As a result of all this, in describing the perturbation series in momentum space, we can continue to use the same Feynman diagrams as for $T = 0$. The only essential difference (apart from differences in coefficients) is the appearance of sums over the discrete frequencies ω_n instead of integrals over the continuous frequency ω in the expressions for the matrix elements.

Before considering specific examples, we derive formulas for the Fourier components of the zeroth-order Green's functions. In Sec. 11, we calculated zeroth-order Green's functions in coordinate space. According to (11.13a) after taking the Fourier transform (14.3) with respect to $\mathbf{r}$, we find that the Green's function of a free fermion has the form

$$\mathscr{G}^{(0)}_{\alpha\beta}(\mathbf{p}, \tau) = -\delta_{\alpha\beta}[1 - n(\mathbf{p})]e^{-[\varepsilon_0(\mathbf{p})-\mu]\tau},$$
$$n(\mathbf{p}) = \{e^{[\varepsilon_0(\mathbf{p})-\mu]/T} + 1\}^{-1}$$

for $\tau > 0$. Substituting this expression into (14.2), we obtain

$$\mathscr{G}^{(0)}_{\alpha\beta}(\mathbf{p}, \omega_n) = -\delta_{\alpha\beta}[1 - n(\mathbf{p})]\int_0^{1/T} e^{i\omega_n\tau-[\varepsilon_0(\mathbf{p})-\mu]\tau}\,d\tau$$

$$= -\frac{\delta_{\alpha\beta}}{i\omega_n - \varepsilon_0(\mathbf{p}) + \mu}[1 - n(\mathbf{p})]\{e^{(2n+1)\pi i-[\varepsilon_0(\mathbf{p})-\mu]/T} - 1\},$$

where $\omega_n = (2n + 1)\pi T$, i.e.,

$$\mathscr{G}^{(0)}_{\alpha\beta}(\mathbf{p}, \omega_n) = \delta_{\alpha\beta}\frac{1}{i\omega_n - \varepsilon_0(\mathbf{p}) + \mu}, \qquad \omega_n = (2n + 1)\pi T. \qquad (14.6) \leftarrow$$

Similar calculations give

$$\mathscr{G}^{(0)}(\mathbf{p}, \omega_n) = \frac{1}{i\omega_n - \varepsilon_0(\mathbf{p}) + \mu}, \qquad \omega_n = 2n\pi T, \qquad (14.7)$$

for bosons, and

$$\mathscr{D}^{(0)}(\mathbf{k}, \omega_n) = -\frac{\omega_0^2(\mathbf{k})}{\omega_n^2 + \omega_0^2(\mathbf{k})}, \qquad \omega_n = 2n\pi T. \qquad (14.8) \leftarrow$$

for phonons. Thus, the zeroth-order Green's functions for fermions and bosons differ only by the "parity" of the frequencies ω_n. The functions (14.6)–(14.8) are obtained from the Green's functions (7.7) and (7.16) by making the substitution $\omega \to i\omega_n$. Later on, we shall show that there is a similar relation between the exact Green's functions for $T = 0$ and $T \neq 0$ (with certain stipulations, of course).

14.2. Examples. As we have seen, the calculation of temperature Green's functions can be carried out by using the technique of Feynman diagrams in momentum space. In doing so, with each line of a given diagram we associate a zeroth-order particle Green's function $\mathscr{G}^{(0)}(\mathbf{p}, \omega_n)$ or a zeroth-order phonon Green's function $\mathscr{D}^{(0)}(\mathbf{k}, \omega_n)$, and with each vertex we associate the quantity $\delta(\Sigma\,\mathbf{p})\delta_{\Sigma\,\omega_n}$ expressing conservation of momentum and conservation of the discrete "frequency" ω_n. Moreover, we integrate over the momenta and sum over the "frequencies" ω_n of all the internal lines.

The actual form of the diagrams and of the associated analytical expressions depends on the form of the interaction. We begin with the case of two-particle interactions.

A. Two-particle interactions. Consider the correction to the Green's function corresponding to diagram (d) of Fig. 33, which was found in Sec. 13 to be

$$-\int d^4z_1\,d^4z_2\,\mathscr{G}^{(0)}_{\alpha\gamma_1}(x - z_1)\mathscr{G}^{(0)}_{\gamma_1\gamma_2}(z_1 - z_2)\mathscr{G}^{(0)}_{\gamma_2\beta}(z_2 - y)\mathscr{V}(z_1 - z_2).$$

Calculating the Fourier components of this expression with respect to the coordinates and "time," we obtain

$$\delta\mathscr{G}^{(1)}(\mathbf{p}, \omega_n) = \frac{1}{2}\int d(\mathbf{x} - \mathbf{y})\int_{-1/T}^{1/T} d(\tau_x - \tau_y)$$
$$\times\ \delta\mathscr{G}^{(1)}(\mathbf{x} - \mathbf{y}, \tau_x - \tau_y)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})+i\omega_n(\tau_x-\tau_y)}.$$

Next, we introduce the Fourier components of the potential $\mathscr{V}(z_1 - z_2)$:

$$\mathscr{V}(\mathbf{r}, \tau) = \frac{T}{(2\pi)^3}\sum_{\omega_n}\int d\mathbf{q}\,e^{i\mathbf{q}\cdot\mathbf{r}-i\omega_n\tau}\mathscr{V}(\mathbf{q}, \omega_n).$$

Since

$$T\sum_{n=-\infty}^{\infty} e^{2n\pi iT\tau} = \delta(\tau),$$
$$\mathscr{V}(\mathbf{q}, \omega_n) = U(\mathbf{q}),$$

we have

$$\delta\mathscr{G}^{(1)}_{\alpha\beta}(\mathbf{p}, \omega_n) = -\frac{1}{2}\left[\frac{T}{(2\pi)^3}\right]^4 \sum_{\substack{\omega_{n1},\,\omega_{n2}\\ \omega_{n3},\,\omega_{n4}}}\int d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4\ \mathscr{G}^{(0)}_{\alpha\gamma_1}(\mathbf{p}_1, \omega_{n1})$$
$$\times\ \mathscr{G}^{(0)}_{\gamma_1\gamma_2}(\mathbf{p}_2, \omega_{n2})\,\mathscr{G}^{(0)}_{\gamma_2\beta}(\mathbf{p}_3, \omega_{n3})\mathscr{V}(\mathbf{q}, \omega_{n4})$$
$$\times\int d(\mathbf{x} - \mathbf{y})\,d\mathbf{z}_1\,d\mathbf{z}_2\,d(\tau_x - \tau_y)\,d\tau_1\,d\tau_2\,e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}e^{i\omega_n(\tau_x-\tau_y)}$$
$$\times\ e^{i\mathbf{p}_1\cdot(\mathbf{x}-\mathbf{z}_1)+i\mathbf{p}_2\cdot(\mathbf{z}_1-\mathbf{z}_2)+i\mathbf{p}_3\cdot(\mathbf{z}_2-\mathbf{y})}e^{-i\omega_{n1}(\tau_x-\tau_1)-i\omega_{n2}(\tau_1-\tau_2)}$$
$$\times\ e^{-i\omega_{n3}(\tau_2-\tau_y)}e^{i\mathbf{q}\cdot(\mathbf{z}_1-\mathbf{z}_2)-i\omega_{n4}(\tau_1-\tau_2)}.$$

Making the change of variables $\mathbf{x} - \mathbf{y} \to \mathbf{x}$, $\tau_x - \tau_y \to \tau$ in the integrals over space and time, we find that

$$\frac{1}{2}\int d\mathbf{x}\, d\mathbf{z}_1\, d\mathbf{z}_2 \int_{-1/T}^{1/T} d\tau \int_0^{1/T}\int_0^{1/T} d\tau_1\, d\tau_2\, e^{i(-\mathbf{p}+\mathbf{p}_1)\cdot\mathbf{x}+i(-\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\cdot\mathbf{z}_1+i(-\mathbf{p}_2+\mathbf{p}_3-\mathbf{q})\cdot\mathbf{z}_2}$$
$$\times\, e^{i(\omega_n-\omega_{n1})\tau+i(\omega_{n1}-\omega_{n2}-\omega_{n4})\tau_1+i(\omega_{n2}-\omega_{n3}+\omega_{n4})\tau_2} e^{i(\mathbf{p}_1-\mathbf{p}_3)\cdot\mathbf{y}+i(-\omega_{n1}+\omega_{n3})\tau_y}$$
$$= \left[\frac{(2\pi)^3}{T}\right]^3 \delta(\mathbf{p}-\mathbf{p}_1)\delta(\mathbf{p}_1-\mathbf{p}_2-\mathbf{q})\delta(\mathbf{p}_2-\mathbf{p}_3+\mathbf{q})$$
$$\times\, \delta_{\omega_n-\omega_{n1}}\delta_{\omega_{n1}-\omega_{n2}-\omega_{n4}}\delta_{\omega_{n2}-\omega_{n3}+\omega_{n4}},$$

which implies that

$$\delta\mathscr{G}^{(1)}_{\alpha\beta}(\mathbf{p},\omega_n) = -\frac{T}{(2\pi)^3}\sum_{\omega_{n1}}\int d\mathbf{p}_1\, \mathscr{G}^{(0)}_{\alpha\gamma_1}(\mathbf{p},\omega_n)$$
$$\times\, \mathscr{G}^{(0)}_{\gamma_1\gamma_2}(\mathbf{p}_1,\omega_{n1})\mathscr{G}^{(0)}_{\gamma_2\beta}(\mathbf{p},\omega_n)\mathscr{V}(\mathbf{p}-\mathbf{p}_1,\omega_n-\omega_{n1}).$$

Substituting the expressions (14.6) and (14.7) for the zeroth-order Green's functions, we finally obtain

$$\delta\mathscr{G}^{(1)}_{\alpha\beta} = -\frac{\delta_{\alpha\beta}}{[i\omega_n-\varepsilon_0(\mathbf{p})+\mu]^2}\frac{T}{(2\pi)^3}\sum_{\omega_{n1}}\int d\mathbf{p}_1\,\frac{\mathscr{V}(\mathbf{p}-\mathbf{p}_1,\omega_n-\omega_{n1})}{i\omega_{n1}-\varepsilon_0(\mathbf{p}_1)+\mu}.$$

Similar calculations for the contribution of diagram (c) of Fig. 33 lead to the result

$$\pm\frac{\delta_{\alpha\beta}}{[i\omega_n-\varepsilon_0(\mathbf{p})+\mu]^2}\mathscr{V}(0,0)(2s+1)\frac{T}{(2\pi)^3}$$
$$\times\sum_{\omega_{n1}}\int d\mathbf{p}_1\,\frac{e^{i\omega_{n1}\tau}}{i\omega_{n1}-\varepsilon_0(\mathbf{p}_1)+\mu}, \tag{14.10}$$

where $\tau \to +0$, and s is the spin of the particle, equal to $\frac{1}{2}$ for fermions and 0 for bosons. Here we have introduced $e^{i\omega_n\tau}$ $(\tau \to +0)$ inside the sum in accordance with the stipulation (made in Sec. 13) that a Green's function in coordinate space with identical time arguments is defined as

$$\mathscr{G}^{(0)}(0,0) = \lim_{\tau\to+0}\mathscr{G}^{(0)}(0,-\tau).$$

The Feynman diagrams corresponding to the expressions (14.9) and (14.10) are shown in Figs. 42(a) and 42(b), respectively. The external lines of these diagrams are labeled with $\mathbf{p}, \omega_n$, the external momentum and frequency. The momenta and frequencies at each vertex satisfy conservation laws, i.e., the sum of all momenta and frequencies "entering" a given vertex equals the sum of all momenta and frequencies "leaving" the vertex.

We now consider the diagram for $\mathscr{G}(\mathbf{p},\omega_n)$ in the kth-order approximation of perturbation theory. Such a diagram has $2k$ vertices, $2k+1$ solid lines and k wavy lines. To calculate the Fourier components, we perform $2k$ integrations over the space and "time" coordinates of the vertices, and one integration over the difference coordinates of the external points. This leads to $2k+1$ quantities of the type $\delta(\Sigma\,\mathbf{p})\delta_{\Sigma\omega_n}$, expressing $2k+1$ conservation laws. It is easy to see that two of these conservation laws express the fact that the external lines have momentum $\mathbf{p}$ and frequency ω_n. The remaining $2k-1$ conservation laws imply that only k integrations and summations actually survive among the $3k-1$ integrations over the momenta and summations over the frequencies of the internal lines (both solid and wavy).

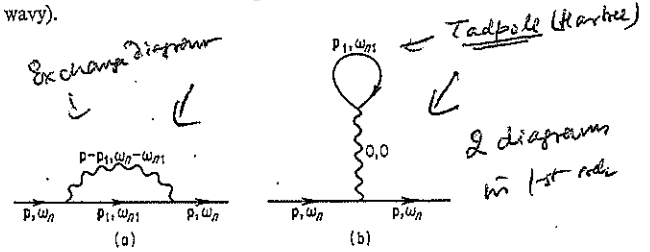


FIGURE 42

Exchange diagram

Tadpole (Hartree)

2 diagrams in 1st order

The general rules for obtaining the expression corresponding to a given diagram for the Green's function are as follows:

1. First, associate momenta and frequencies with the lines of the diagram in such a way that the external lines have the external momentum and frequency, while the momenta and frequencies of the internal lines satisfy the conservation laws $\Sigma\,\mathbf{p}' = 0$, $\Sigma\,\omega'_n = 0$, where the frequencies of Bose lines are always even $[\omega_n = 2n\pi T]$ and those of Fermi lines are odd $[\omega_n = (2n+1)\pi T]$.
2. Integrate and sum over the independent momenta and frequencies.
3. With each solid internal line (of momentum $\mathbf{p}'$ and frequency ω'_n) associate a quantity
$$\frac{1}{i\omega'_n - \varepsilon_0(\mathbf{p}') + \mu},$$
and with each wavy line (of momentum $\mathbf{q}'$ and frequency ω'_n) associate a quantity
$$\mathscr{V}(\mathbf{q},\omega'_n) \equiv U(\mathbf{q}).$$
4. With both external lines (of momentum $\mathbf{p}$ and frequency ω_n) associate a quantity
$$\frac{\delta_{\alpha\beta}}{[i\omega_n - \varepsilon_0(\mathbf{p}) + \mu]^2}$$
5. Multiply the resulting expression by
$$(-1)^k\frac{T^k}{(2\pi)^{3k}}(2s+1)^F(\mp 1)^F,$$
where F is the number of closed loops formed by particle lines.

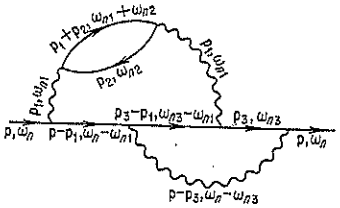
Using these rules, it is not hard to write down the correction corresponding to an arbitrarily complicated diagram. For example, the correction corresponding to the diagram shown in Fig. 43 equals

$$\pm \frac{\delta_{\alpha\beta}}{[i\omega - \varepsilon_0(\mathbf{p}) + \mu]^2} \frac{T^3}{(2\pi)^9} (2s+1) \sum_{\omega_1, \omega_2, \omega_3} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3$$
$$\times \frac{1}{i(\omega - \omega_1) - \varepsilon_0(\mathbf{p} - \mathbf{p}_1) + \mu} \frac{1}{i(\omega_3 - \omega_1) - \varepsilon_0(\mathbf{p}_3 - \mathbf{p}_1) + \mu}$$
$$\times \frac{1}{i\omega_3 - \varepsilon_0(\mathbf{p}_3) + \mu} \frac{1}{i\omega_2 - \varepsilon_0(\mathbf{p}_2) + \mu} \frac{1}{i(\omega_1 + \omega_2) - \varepsilon_0(\mathbf{p}_1 + \mathbf{p}_2) + \mu}$$
$$\times [U(\mathbf{p}_1)]^2 U(\mathbf{p} - \mathbf{p}_3),$$

where

$$\omega_1 = 2n\pi T, \qquad \omega_2, \omega_3 = (2n+1)\pi T.$$

Next, we consider the other version of the diagram technique for the case of two-particle interactions. To go over to the Fourier representation in the corresponding expressions of Sec. 13, it is convenient to use the following formal device: Earlier, we introduced the quantity $\mathcal{T}^{(0)}_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}(z_1, z_2; z_3, z_4)$. This quantity depends on four "times" τ_i, where each τ_i varies in the interval from 0 to $1/T$. We continue the function $\mathcal{T}^{(0)}$ onto the interval from $-1/T$ to $1/T$ by using relations like (11.8) for the $\mathcal{G}$-function, i.e.,



3rd order

FIGURE 43

$$\mathcal{T}^{(0)}(\tau_1 < 0, \tau_2; \tau_3, \tau_4) = \mp \mathcal{T}^{(0)}(\tau_1 + \frac{1}{T}, \tau_2; \tau_3, \tau_4),$$

and similarly for τ_2, τ_3, τ_4. Then we define the Fourier components with respect to the τ_i by

$$\frac{1}{16} \int_{-1/T}^{1/T} \cdots \int_{-1/T}^{1/T} d\tau_1 \cdots d\tau_4\, e^{i(\omega_1\tau_1 + \omega_2\tau_2 - \omega_3\tau_3 - \omega_4\tau_4)} \mathcal{T}^{(0)}(\tau_1, \tau_2; \tau_3, \tau_4).$$

Obviously, all four frequencies are "odd" in the case of Fermi statistics, and "even" in case of Bose statistics.

Moreover, we note that since $\mathcal{T}^{(0)}(z_1, z_2; z_3, z_4)$ is by definition a function only of the coordinate and "time" differences, the Fourier components of $\mathcal{T}^{(0)}$ with respect to the space and time variables contain a δ-function $\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$ involving a sum of momenta and a Kronecker delta $\delta_{\omega_1+\omega_2-\omega_3-\omega_4}$ involving a sum of frequencies. Therefore, we define the Fourier component of $\mathcal{T}^{(0)}$ directly as

$$\frac{(2\pi)^3}{T} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta_{\omega_1+\omega_2-\omega_3-\omega_4} \mathcal{T}^{(0)}(\mathbf{p}_1, \omega_1, \mathbf{p}_2, \omega_2; \mathbf{p}_3, \omega_3, \mathbf{p}_4, \omega_4)$$
$$= \frac{1}{16} \int_{-1/T}^{1/T} \cdots \int_{-1/T}^{1/T} d\tau_1 \cdots d\tau_4 \int d\mathbf{r}_1 \cdots d\mathbf{r}_4 \tag{14.11}$$
$$\times e^{-i(\mathbf{p}_1\cdot\mathbf{r}_1 + \mathbf{p}_2\cdot\mathbf{r}_2 - \mathbf{p}_3\cdot\mathbf{r}_3 - \mathbf{p}_4\cdot\mathbf{r}_4) + i(\omega_1\tau_1 + \omega_2\tau_2 - \omega_3\tau_3 - \omega_4\tau_4)} \mathcal{T}^{(0)}(z_1, z_2; z_3, z_4).$$

For example, suppose we carry out the Fourier transformation of the expression for the first-order correction corresponding to diagram (b) of Fig. 35 (see p. 116). Then, after simple calculations, we obtain

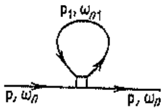


FIGURE 44

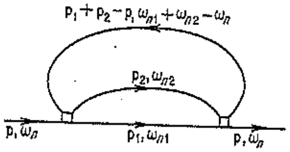


FIGURE 45

$$-\frac{1}{[i\omega - \varepsilon_0(\mathbf{p}) + \mu]^2} \frac{T}{(2\pi)^3}$$
$$\times \sum_{\omega_1} \int d\mathbf{p}_1\, \mathcal{T}^{(0)}_{\alpha\gamma,\gamma\beta}(\mathbf{p}, \omega, \mathbf{p}_1, \omega_1; \mathbf{p}_1, \omega_1, \mathbf{p}, \omega) \frac{1}{i\omega_1 - \varepsilon_0(\mathbf{p}_1) + \mu},$$

which corresponds to the diagram shown in Fig. 44. Similar calculations for the correction corresponding to the diagram shown in Fig. 45 lead to the formula

$$\frac{1}{2} \frac{1}{[i\omega - \varepsilon_0(\mathbf{p}) + \mu]^2} \frac{T^2}{(2\pi)^6} \sum_{\omega_1, \omega_2} \int d\mathbf{p}_1\, d\mathbf{p}_2\, \mathcal{T}^{(0)}_{\alpha\gamma_1,\gamma_2\gamma_3}(p, p_1 + p_2 - p; p_1, p_2)$$
$$\times \frac{1}{i\omega_1 - \varepsilon_0(\mathbf{p}_1) + \mu} \frac{1}{i\omega_2 - \varepsilon_0(\mathbf{p}_2) + \mu} \frac{1}{i(\omega_1 + \omega_2 - \omega) - \varepsilon_0(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}) + \mu}$$
$$\times \mathcal{T}^{(0)}_{\gamma_2\gamma_3,\gamma_1\beta}(p_1, p_2; p_1 + p_2 - p, p),$$

where we have used the four-dimensional notation $p = (\mathbf{p}, \omega_n)$.

The nth-order diagram for the $\mathcal{G}$-function contains n squares (vertices) and $2n + 1$ lines, where the $2n + 1$ internal lines obey the conservation laws $\Sigma\, \mathbf{p}' = 0, \Sigma\, \omega_n' = 0$ at the vertices. It is easy to see that there are a total of n independent integrations and summations over the momenta and frequencies