Physics 220- Fall 2017

Theory of Many Body Physics

Homework 4
27 October, 2017

The first nine problems refer to the notes from Fetter and Walecka on the degenerate electron gas below (pages 21-31 of the book- a copy has been given to all students.)

1. Show that Eq (3.9) is correct.
2. Derive Eq(3.19) fully, making sure of the cancellation of the q=0 term.
3. Verify the scaling with density in 3.24.
4. Show that

$$\alpha r_s a_0 k_F = 1$$

   where $\alpha = [4/(9\pi)]^{1/3} \sim .52.$ and prove that in the Fermi gas ground state (determinant of equal up and down electrons)

$$\langle N \rangle = V k_F^3/(3\pi^2),$$

   and kinetic energy

$$E^{(0)} = \langle T \rangle = 3/5 N \varepsilon_F.$$

5. Consider a spin unbalanced population where

$$N_\uparrow = \frac{1}{2}(1+x)N, \quad N_\downarrow = \frac{1}{2}(1-x)N,$$

   where $-1 \leq x \leq 1$ is the fractional magnetization. Show that in such a ground state the kinetic energy

$$E^{(0)} = \left(E^{(0)}\right)_{x=0} \times \frac{1}{2}\left((1+x)^{5/3} + (1-x)^{5/3}\right),$$

   and hence verify that the minimum kinetic energy is reached at $x = 0$.

6. Verify Eq(3.34) and the next equation.
7. (Optional) Either analytically or by monte carlo integration verify Eq(3.35).

8. Using the partially magnetized state above show that the exchange energy Eq(3.36) generalizes in this state to

$$E^{(1)} = -\frac{e^2}{2a_0} N \frac{.916}{r_s} \frac{1}{2} \left((1+x)^{4/3} + (1-x)^{4/3} \right),$$

by using scaling. (This means you are allowed to assume the result at $x = 0$ and start from that.)

9. Using the total energy $E^{(0)} + E^{(1)}$ as a function of x, show that the fully ferromagnetic state $x = 1$ is the ground state for $r_s >\sim 4.54$, while the paramagnet is the ground state for higher densities. What happens to the case of partial polarization, i.e. is $|x| < 1$ ever a solution?

10. Using the Dyson representation of the BCS state in reverse, find the BCS wave function given the Dyson form

$$|\Psi\rangle = (\eta^\dagger)^{N/2} |vac\rangle,$$

where

$$\eta^\dagger = \sum_{ij} e^{-|i-j|} f_{i\uparrow}^\dagger f_{j\downarrow}^\dagger |0\rangle,$$

where i, j are the dimensionless coordinates of particles on a 1-d ring. You will need $\sum_r e^{-|r|+irq}$ the Fourier transform of the pair wave function. This can be calculated by approximating the sum by the integral.