Physics 220- Fall 2017

Theory of Many Body Physics

Homework-3: Heisenberg Model Notes and problems
13 October, 2017

§.1 Its is very useful to review the angular momentum theory for N spins, as a model for second quantization. We can see that N spin-s particles on a lattice is in essence a Bose gas on a lattice, with extra constraint of hard core.

Let us consider

$$H = -J \sum_{<ij>} \vec{S}_i . \vec{S}_j - \vec{b} . \sum_j \vec{S}_j , \tag{1}$$

where the symbol $\sum_{<ij>} A_{ij}$ implies summing A_{ij} over each bond once and only once. This can be also rewritten as $\frac{1}{2}\sum_{ij} J_{ij} A_{ij}$ where $J_{ij} = \delta_{|\vec{r}_i - \vec{r}_j|, a_0}$, and the factor of half accounts for double counting. The case $J > 0$ is ferromagnetic and easy, the case $J < 0$ is antiferromagnetic and usually hard.

The spins at different sites are assumed to commute and at a single site they satisfy the usual relations

$$[S_i^\alpha, S_j^\beta] = i\hbar \delta_{ij} S_i^\gamma \, \epsilon^{\alpha\beta\gamma} \tag{2}$$

We can recombine spins into lowering and raising operators and write

$$S_i^+ = S_i^x + i S_i^y, \;\; S_i^- = S_i^x - i S_i^y . \tag{3}$$

We also write a number operator $\hat{n}_i$, which counts the number of spin deviations from the ground state $|0>=|\downarrow>$, as

$$\hat{n}_i = S_i^z + S. \tag{4}$$

Here S is the length of the spin giving the dimensionality of the spin operators as $2S+1$, thus $S = 1/2$ has a 2×2 matrix representation at each site. The commutators read

$$\begin{aligned} [S_i^+, \hat{n}_j] &= \delta_{ij} \, S_i^+ \\ [S_i^-, \hat{n}_j] &= -\delta_{ij} \, S_i^- \\ [S_i^-, S_j^+] &= -2\delta_{ij} \, S^z \\ &= 2S(1 - \hat{n}_i/(S)). \end{aligned} \tag{5}$$

We may regard each spin reversal from the vacuum state as a particle. Thus the eigenvalue n_i of the corresponding operator gives the number of particles/ The restriction of hard-core means $0 \leq n_i \leq 2S$, i.e. no more than $2S$ particles.

Redefining $S_i^- = \sqrt{2S} b_i$ and $S_i^+ = \sqrt{2S} b_i^\dagger$, it is easily verified that in the limit of large S, these operators become standard Boson operators. Thus the particles for large S are regular Bosons, whereas for finite S they are Bosons with a kinematical hardcore constraint.

§.2 For the Ferromagnet, the ground state is assumed as

$$|0>=|\downarrow\downarrow\ldots\downarrow> \tag{6}$$

- Problem (1) Using $S_i^-|0>=0$ show that $H|0>=E_0|0>$, where

  $$E_0 = -NJS^2Z/2,$$

  with Z as the coordination number $Z = 2, 4, 6$ for the 1-d, 2-d and 3-d simple (chain, square, cube) lattices.

- Problem (2). Argue that E_0 is the absolute ground state. This needs the observation that each bond in the Hamiltonian is in its ground state in the state $|0>$.

- Problem (3) For $J = -|J|$, the antiferromagnet, consider *only* the 1-d chain with an even number of sites. Here it is tempting to think of a state due to Louis Neel:

  $$|Neel>=|\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow>, \tag{7}$$

  as a potential ground state. Verify that $|Neel>$ is *not* an eigenstate of H.

- Problem (4) In the above problem calculate the variational energy

  $$< Neel|H|Neel >, \tag{8}$$

  and show that the answer is

  $$E_{Neel} = -N|J|S^2. \tag{9}$$

  Argue that this cannot be the ground state by calculating the minimum energy of a pair of spins $-|J|\vec{S}_1.\vec{S}_2$, and showing that this is lower than what E_{Neel} provides for a pair. (This leads to the idea of quantum fluctuations in antiferromagnets.)

- Problem (5) Consider a state (Ferromagnet in 1-d with N sites)

$$|k> = \frac{1}{\sqrt{N}} \sum_i e^{ikr_i} \; S_i^+ |0> \; . \tag{10}$$

a) Show that $|k>$ is an eigenstate of H

b) Calculate the spin wave energy in the state $k>$ as given in class

$$\begin{aligned} E_k &= E_0 + \hbar\omega_k \\ \hbar\omega_k &= 2JS^2(1 - \cos(ka_0)). \end{aligned} \tag{11}$$