Physics 220- Fall 2017

Manybody Physics

Test-1
100 points, Time 90 mins
20 October, 2017

SOLUTION

We will abbreviate anti-commutation-relations by ACR, and commutation-relations by CR.

1. *Using Fermionic anticommutators show that*

$$f_{\alpha\sigma} f_{\alpha\sigma} = 0,$$

where α is any single particle label, and also show that

$$n^2_{\alpha\sigma} = n_{\alpha\sigma},$$

where $n_{\alpha\sigma} = f^\dagger_{\alpha\sigma} f_{\alpha\sigma}$. ... *[10]*

The first part follows from $\{f_{\alpha\sigma}, f_{\alpha\sigma}\} = 0 = 2 f_{\alpha\sigma} f_{\alpha\sigma}$.

For the second part we write

$$n^2_{\alpha\sigma} = f^\dagger_{\alpha\sigma} f_{\alpha\sigma} f^\dagger_{\alpha\sigma} f_{\alpha\sigma} = f^\dagger_{\alpha\sigma} f_{\alpha\sigma} - f^\dagger_{\alpha\sigma} f^\dagger_{\alpha\sigma} f_{\alpha\sigma} f_{\alpha\sigma} = f^\dagger_{\alpha\sigma} f_{\alpha\sigma},$$

the key point is that we used the ACR to replace $f_{\alpha\sigma} f^\dagger_{\alpha\sigma} = 1 - f^\dagger_{\alpha\sigma} f_{\alpha\sigma}$, and then used $f_{\alpha\sigma} f_{\alpha\sigma} = 0$.

2. *The Hubbard model for Fermions is given as:*

$$H = -\sum_{ij} t_{ij} C^\dagger_{i\sigma} C_{j\sigma} + U \sum_i n_{i\uparrow} n_{\downarrow}.$$

Show that the interaction term can be written as

$$U \sum_i \left(\frac{1}{4} \rho_i^2 - (S_i^z)^2 \right),$$

where the density $\rho_i = \sum_\sigma n_{i\sigma}$ and spin density $S_i^z = \frac{1}{2} \sum_\sigma \sigma \, n_{i\sigma}$, and $\sigma = \pm 1$. ... *[20]*

This is done by writing

$$\rho_i^2 = (n_{i\uparrow} + n_{i\downarrow})^2 = n_{i\uparrow} + n_{i\downarrow} + 2 n_{i\uparrow} n_{i\downarrow},$$

and

$$4(S_i^z)^2 = (n_{i\uparrow} - n_{i\downarrow})^2 = n_{i\uparrow} + n_{i\downarrow} - 2 n_{i\uparrow} n_{i\downarrow}.$$

Subtracting the two equation and dividing by 4 gives the required result.

3. *For two spin 1 particles interacting with the Hamiltonian*

$$H = |J| \vec{S}_1.\vec{S}_2,$$

find all *the eigenvalues and degeneracies.* ... *[20]*

{*Comment: No need to write the eigenfunctions.* }

We write

$$\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2,$$

and verify that

$$\vec{S}_{tot} \times \vec{S}_{tot} = \vec{S}_1 \times \vec{S}_1 + \vec{S}_2 \times \vec{S}_2 + (\vec{S}_1 \times \vec{S}_2 + \vec{S}_2 \times \vec{S}_1).$$

On using $\vec{S}_j \times \vec{S}_j = i\vec{S}_j$ and $\vec{S}_1 \times \vec{S}_2 = -\vec{S}_2 \times \vec{S}_1$ the above expression reduces to

$$\vec{S}_{tot} \times \vec{S}_{tot} = i\vec{S}_{tot},$$

and therefore we conclude that $\vec{S}_{tot}$ is an angular momentum operator. (This extended discussion is to help those who may have forgotten this basic result from QM). The eigenvalues of $\vec{S}_{tot}.\vec{S}_{tot}$ are $S_{tot}(S_{tot}+1)$, where

$$0 \leq S_{tot} \leq 2S,$$

with $S = 1$ from the problem. We also know that each value S_{tot} has a $2S_{tot} + 1$ fold degeneracy.

Now

$$\vec{S}_1.\vec{S}_2 = \frac{1}{2}\left(\vec{S}_{tot}^2 - \vec{S}_1.\vec{S}_1 - \vec{S}_2.\vec{S}_2\right),$$

and using the result proved above

$$\left(\vec{S}_1.\vec{S}_2\right)_{EV} = \frac{1}{2} S_{tot}(S_{tot}+1) - 2.$$

The eigenvalues of $H/|J|$ and their degeneracy are thus $-2(1), -1(3), 1(5)$. This accounts for 9 eigenstates, this is clearly the right number for two spin 1 particles.

4. *Using standard Bosonic commutators for b_k and $b_k^\dagger$, show that*

$$H_0 b_k^\dagger b_p^\dagger |0> = (\varepsilon_k + \varepsilon_p) b_k^\dagger b_p^\dagger |0>,$$

where $H_0 = \sum_\alpha \varepsilon_\alpha b_\alpha^\dagger b_\alpha$. ... *[25]*

We first establish the simple result

$$[H_0, b_k^\dagger] = \varepsilon_k b_k^\dagger,$$

and $H_0|0> = 0$ from the basic CR's of Bosons. Using these we rewrite

$$H_0 b_k^\dagger b_p^\dagger |0> = [H_0, b_k^\dagger b_p^\dagger]|0> = [H_0, b_k^\dagger] b_p^\dagger |0> + b_k^\dagger [H_0, b_p^\dagger]|0> = (\varepsilon_k + \varepsilon_p) b_k^\dagger b_p^\dagger |0> .$$

5. *We saw that a Majorana fermion* Φ *has the property* $\Phi^2 = 1$. *Consider three flavors of Majorana fermions* Φ_x, Φ_y, Φ_z, *which mutually anticommute. Thus we write*

$$\{\Phi_a, \Phi_b\} = 2\delta_{ab}$$

with $a = x, y, z$. *From these we construct three new objects*

$$\tau_x = i\Phi_y\Phi_z, \ \tau_y = i\Phi_z\Phi_x, \ \ \tau_z = i\Phi_x\Phi_y.$$

Show that τ_a *satisfy all the properties of Pauli matrices, namely*

$$\tau_a\tau_b = i\varepsilon_{abc}\tau_c + \delta_{ab},$$

for any choice of a, b, c, *and with* ε_{abc} *as the usual antisymmetric tensor* $\varepsilon_{xyz} = 1 = -\varepsilon_{yxz}$ *etc.* *...[25]*

Using symmetry we need to examine only two cases: $a = b = x$ and $a = x, b = y, c = z$, for these cases we need to show

$$\tau_x\tau_x = 1,$$

and

$$\tau_x\tau_y = i\tau_z.$$

To check the first we plug in the Majorana representation

$$\tau_x\tau_x = -\Phi_y\Phi_z\Phi_y\Phi_z = +\Phi_y^2\Phi_z^2 = 1.$$

where we used the ACR of Φ_y and Φ_z and also $\Phi_y^2 = 1 = \Phi_z^2$.

To check the second we plug in the Majorana representation again

$$\tau_x\tau_y = -\Phi_y\Phi_z\Phi_z\Phi_x = -\Phi_y\Phi_x = \Phi_x\Phi_y = -i\tau_z.$$

Clearly I missed a minus sign in the definitions since I should have a $i\tau_z$ instead. We can correct this minor error by redefining

$$\tau_x = -i\Phi_y\Phi_z, \ \tau_y = -i\Phi_z\Phi_x, \ \ \tau_z = -i\Phi_x\Phi_y.$$

instead of the original ones. The negative sign is fixed with this new definition since the left hand side is unchanged and the right hand side picks up an extra minus sign that we want.