a superconductor, we must reexpress this electron state in terms of the appro-
riate excitations, the $\gamma_{\mathbf{k}}$, using (2-68). The result is

$$c^*_{\mathbf{k}\uparrow} = u_{\mathbf{k}}\gamma^*_{e\mathbf{k}0} + v^*_{\mathbf{k}}\gamma_{h\mathbf{k}1} \tag{2-75}$$

f the superconductor is in its ground state, the second term gives zero, and the process contributes a current proportional to $|u_{\mathbf{k}}|^2|T_{\mathbf{jk}}|^2$. The physical significance of the factor $|u_{\mathbf{k}}|^2$ is that it is the probability that the state $\mathbf{k}$ is *not* occupied in the BCS function, and hence is able to receive an incoming electron. Thus, it appears on the face of it that the tunneling current will depend on the nature of the superconducting ground state as well as on the density of available excited states; but this turns out not to be true. As is evident from Fig. 2-5, there is another state $\mathbf{k}'$ having exactly the same energy $E_{\mathbf{k}'} = E_{\mathbf{k}}$, but with $\xi_{\mathbf{k}'} = -\xi_{\mathbf{k}}$. Using (2-75), with $\mathbf{k}$ replaced by $\mathbf{k}'$ we see that tunneling into $\mathbf{k}'$ contributes a current proportional to $|u_{\mathbf{k}'}|^2|T_{\mathbf{jk}'}|^2 = |v_{\mathbf{k}}|^2|T_{\mathbf{jk}'}|^2$, since $|u(-\xi)| = |v(\xi)|$. Making the reasonable assumption that the two matrix elements are nearly equal, since $\mathbf{k}$ and $\mathbf{k}'$ are both near the same point on the Fermi surface, the total current from these two channels is proportional to $(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2)|T_{\mathbf{jk}}|^2 = |T_{\mathbf{jk}}|^2$, and the characteristic coherence factors of the superconducting wavefunction, $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, have dropped out. If we now generalize to finite temperatures, so that the quasi-particle occupation numbers $f_{\mathbf{k}}$ are nonzero, both terms of (2-75) contribute, the first as $(1 - f_{\mathbf{k}})$, the second as $f_{\mathbf{k}}$. Again, when the degenerate channels are combined, the current is simply proportional to $|T_{\mathbf{jk}}|^2$.

### 2-8.1 The Semiconductor Model

This disappearance of the coherence factors $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ makes it possible and convenient to reexpress the computation of the tunneling current in what is often called the "semiconductor model." In this method, the normal metal is represented in the familiar elementary way as a continuous distribution of independent-particle energy states with density $N(0)$, including energies below as well as above the Fermi level. The superconductor is represented by an ordinary semiconductor with a density of independent-particle states obtained from Fig. 2-4 by adding its reflection on the negative-energy side of the chemical potential, so that it will reduce properly to the normal-metal density of states as $\Delta \to 0$. At $T = 0$, all states up to $\mu$ are filled; for $T > 0$, the occupation numbers are given by the Fermi function. It is worth noting that $f_{\mathbf{k}}$ now runs from 0 to 1, whereas in our previous convention, $f_{\mathbf{k}}$ ranged only from 0 to $\frac{1}{2}$ since $E_{\mathbf{k}} \geq 0$. This difference reflects the fact that in the present model $f_{\mathbf{k}}$ measures a departure from the vacuum, whereas in the previous "excitation representation" it measured a departure from the ground state of the system.



With this model, tunneling transitions are all "horizontal," that is, they occur at constant energy after adjusting the relative levels of $\mu$ in the two metals to account for the applied potential difference $eV$. This property facilitates summing up all contributions to the current in an elementary way, since the various parallel channels noted above do not have to be considered anew in each case. Because this scheme so greatly simplifies the computations, we shall use it here to work out the tunneling characteristics of various types of junctions, and simply refer the reader to more detailed treatments which are available in the literature.[1] It should be borne in mind, however, that this technique to some extent *over*simplifies. It is sound to treat the normal metal in this way, but it is less safe to conceal the mixing of hole and electron states which is present in the superconducting state even at $T = 0$. Although our argument above for simply adding the currents from the two degenerate channels is valid for the usual case, there can be an interference effect between them which causes an oscillatory variation of the tunnel current with voltage or sample thickness known as the Tomasch effect.[2] This comment illustrates the need for caution in using the semiconductor model. Of course, this model also is inadequate for dealing with processes in which the condensed pairs play a role, since the ground state does not appear in the energy-level diagram.

Within the independent-particle approximation, the tunneling current from metal 1 to metal 2 can be written as

$$I_{1\to 2} = A\int_{-\infty}^{\infty} |T|^2 N_1(E) f(E) N_2(E + eV)[1 - f(E + eV)]\, dE$$

where $V$ is the applied voltage, $eV$ is the resulting difference in the chemical potential across the junction, and $N(E)$ is the appropriate normal or superconducting density of states. The factors $N_1 f$ and $N_2(1 - f)$ give the numbers of occupied initial states and of available (i.e., empty) final states in unit energy interval. This expression assumes a constant tunneling-matrix element $T$; $A$ is a constant of proportionality. Subtracting the reverse current, the net current is

$$I = A|T|^2 \int_{-\infty}^{\infty} N_1(E) N_2(E + eV)[f(E) - f(E + eV)]\, dE \tag{2-76}$$

We shall now use this expression to treat a number of important cases.

[1] A particularly explicit discussion of the contributions of the various channels is given by M. Tinkham, *Phys. Rev.* **B6**, 1747 (1972). Earlier treatments and reviews have been given by M. H. Cohen, L. M. Falicov, and J. C. Phillips, *Phys. Rev. Letters* **8**, 316 (1962); D. H. Douglass, Jr., and L. M. Falicov, in C. J. Gorter (ed.), "Progress in Low Temperature Physics," vol. 4, p. 97, North-Holland, Amsterdam, 1964; W. L. McMillan and J. M. Rowell, in R. D. Parks (ed.), "Superconductivity," vol. 1, chap. 11, Marcel Dekker, New York, 1969.

[2] W. J. Tomasch, *Phys. Rev. Letters* **15**, 672 (1965); **16**, 16 (1966).

### 2-8.2 Normal-Normal Tunneling

If both metals are normal, (2-76) becomes

$$I_{nn} = A\,|T|^2 N_1(0)N_2(0)\int_{-\infty}^{\infty}[f(E)-f(E+eV)]\,dE$$
$$= A\,|T|^2 N_1(0)N_2(0)eV \equiv G_{nn}V \qquad (2\text{-}77)$$

so that the junction is "ohmic," i.e., it has a well-defined conductance $G_{nn}$, independent of $V$. Note that it is also independent of the temperature.

To help reduce any lingering confusion about the relation of this semiconductor, or independent-particle, scheme to the elementary excitation scheme, let us indicate how this simple case would have been treated in the other framework. First, at $T = 0$, all $f_k = 0$, and there are no excitations present, both metals being in their Fermi sea ground states. Thus any tunneling process must involve creating two excitations, a hole in one metal and an electron in the other, the sum of the two excitation energies being $eV$, as given by (2-72). The resulting current is

$$I = A\,|T|^2\int_0^{eV} N_1(E)N_2(eV-E)\,dE$$
$$= A\,|T|^2 N_1(0)N_2(0)eV$$

exactly as found in (2-77). For $T > 0$, the current from this process is reduced by the excitations already present, which block final states, but this effect is canceled by the extra current from the tunneling of the excitations, leading to a temperature-independent result.

### 2-8.3 Normal-Superconductor Tunneling

A more interesting case arises if one metal is superconducting. Then (2-76) becomes

$$I_{ns} = A\,|T|^2 N_2(0)\int_{-\infty}^{\infty} N_{1s}(E)[f(E)-f(E+eV)]\,dE$$
$$= \frac{G_{nn}}{e}\int_{-\infty}^{\infty}\frac{N_{1s}(E)}{N_1(0)}[f(E)-f(E+eV)]\,dE \qquad (2\text{-}78)$$

In general, numerical means are required to evaluate this expression for the BCS density of states and thus to allow quantitative comparison with experiment, although the qualitative behavior is easily sketched. As indicated in Fig. 2-6*a*, at $T = 0$ there is no tunneling current until $e|V| \geq \Delta$, since the chemical-potential difference must provide enough energy to create an excitation in the superconductor. The magnitude of the current is independent of the sign of $V$ because hole and electron excitations have equal energies. For $T > 0$, the energy of excitations



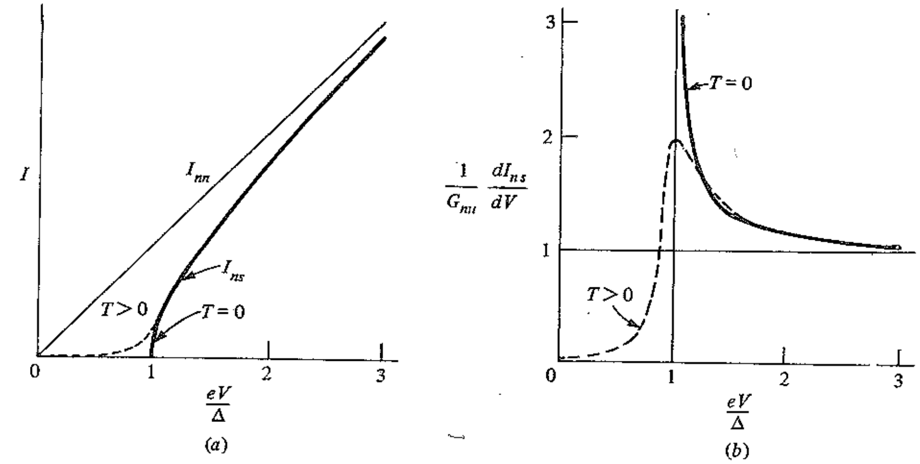


FIGURE 2-6
Characteristics of normal-superconductor tunnel junctions. (*a*) *I-V* characteristic. (*b*) Differential conductance. Solid curves refer to $T = 0$; dashed curves, to a finite temperature.

already present allows them to tunnel at lower voltages, giving an exponential tail of the current in the region below $eV = \Delta$.

A more direct comparison of theory and experiment can be made if one considers the differential conductance $dI/dV$ as a function of $V$. From (2-78)

$$G_{ns} = \frac{dI_{ns}}{dV} = G_{nn}\int_{-\infty}^{\infty}\frac{N_{1s}(E)}{N_1(0)}\left[-\frac{\partial f(E+eV)}{\partial(eV)}\right]dE \qquad (2\text{-}79)$$

Since $-[\partial f(E+eV)/\partial(eV)]$ is a bell-shaped weighting function peaked at $E = -eV$, with width $\sim kT$ and unit area under the curve, it is clear that as $kT \to 0$, this approaches

$$G_{ns}\bigg|_{T=0} = \frac{dI_{ns}}{dV}\bigg|_{T=0} = G_{nn}\frac{N_{1s}(e|V|)}{N_1(0)} \qquad (2\text{-}80)$$

Thus, in the low-temperature limit, the differential conductance measures directly the density of states. At finite temperatures, as shown in Fig. 2-6*b*, the conductance measures a density of states smeared by $\sim kT$ in energy, due to the width of the weighting function. Because this function has exponential "skirts," it turns out that the differential conductance at $V = 0$ is related exponentially to the width of the gap. In the limit $kT \ll \Delta$, this relation reduces to

$$\frac{G_{ns}}{G_{nn}}\bigg|_{V=0} = \left(\frac{2\pi\Delta}{kT}\right)^{1/2} e^{-\Delta/kT} \qquad (2\text{-}81)$$