Quantum Integrable systems

“Where do they all come from?”*

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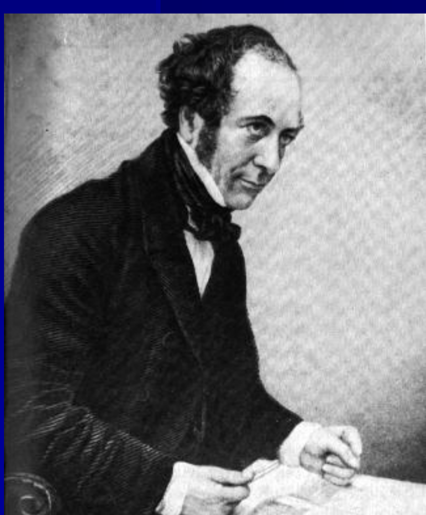
*Eleanor Rigby

PLAN

- Introduction
  - Classical Integrable systems: (Role of higher conservation laws or dynamical symmetries) “Solitons, classical and quantum”
    - Runge Lenz vector as example of dynamical symmetry.
    - KdV Equation, Non Linear Schroedinger Eqn,
  - Quantum Integrable systems:
    - Sine Gordon theory, Massive Thirring model, δ function Bose/Fermi gas, Calogero Sutherland systems. Lax Equations
    - Lattice models: XXZ, XYZ spin chains, Baxter’s eqns, Commuting transfer matrices
- Recent work on energy level statistics/ level crossings:
  - Heilmann-Lieb, Yuzbashyan-Altshuler-Shastry
- A recent preprint (SS Jan ‘05)
- Open Questions

Chris Eilbeck, Alwyn Scott and Martin Kruskal looking for a soliton

The Scott Russell Aqueduct on the Union Canal Edinborough, Scotland



``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation''

John Scott Russell (1845)

Classical Integrable systems: (soliton bearing systems)

Number of degrees of freedom = No of invariants in involution Toda lattice and hence to Korteweg de Vries(KdV) theory

Toda Lattice

$$H = \sum_j \frac{p_j^2}{2m} + g \sum_i \exp(x_j - x_{j+1})$$

KdV: Shallow water waves $u \equiv u(x,t)$ is the height of the water surface

$$u_t = -6uu_x + u_{xxx}$$

Special solitonic solutions: (v=v(a)) Single soliton:

$$u(x,t) = \frac{a}{\cosh^2(x - vt)}$$

Similarly multisoliton solutions exist: Scatter like particles

Secret of solvability:

Higher conservation laws: Lax formulation (Historically GGKM had a more tortured path) $A = -4\partial_x^3 + 3(u\partial_x + \partial_x u)$

$$L = -\partial_x^2 + u(x,t)$$

P Laxs eqn (1968) for Lax pair

$$\partial_t L = u_t = [L, A]$$

Hence eigenvalues of L are constants of motion.

$I_0 = \int dxu$, $I_1 = \int dx[\frac{u^2}{2} + u^3]$,

Similarly Non Linear Schroedinger Equation:

$$i\hbar\partial_t\psi(x,t) = -\partial_x^2\psi(x,t) + g|\psi(x,t)|^2\psi(x,t)$$

Bose gas, Hartree Fock theory, Fiber optics: Hasegawa, it even makes $s with AT&T (maybe)!!

“Bell Labs researchers set new soliton transmission record

“AT&T Bell Laboratories scientists have demonstrated error-free transmission of solitons (light pulses that maintain their shape over long distances) at 5 gigabits (billion bits) per second over 15,000 kilometers and at 10 gigabits over 11,000 kilometers. A research team led by physicist Linn Mollenauer, of the Bell Labs Photonic Circuits Research department used time-division multiplexing (interleaving bits of information from one stream of data into the spaces of another) to upgrade a 2.5-gigabit signal to 5 gigabits and then used wavelength-division multiplexing (transmitting data on two wavelengths, or colors, of light) to reach 10 gigabits.

Kepler problem / Hydrogen atom

Runge Lenz Vector is conserved:

$$\vec{R} = \vec{p} \wedge \vec{L} - \frac{me^2 \vec{r}}{|\vec{r}|}$$

Example of dynamical symmetry: leads to permanent degeneracy in spectrum of Hydrogen, SO(4) symmetry in momentum space.

Quantum Integrable Systems:

Quantum sine Gordon theory: equivalence to massive Thirring model – (S Coleman 1975) Dashen Hasslacher Neveau semiclassical quantization of Sine Gordon- created considerable excitement and interest:

Alan Luther: Anistropic Heisenberg model is equivalent to Sine Gordon. Anisotropic Heisenberg model is a DISCRETE space problem of “spins” but due to a simple Jordan Wigner Transformation + continuum limit becomes Massive Thirring model- which is solvable by Bethe’s Ansatz!! (integrability)

$$
\begin{aligned}
H &= \sum_i (J_x\sigma^x(i)\sigma^x(i+1) + J_y\sigma^y(i)\sigma^y(i+1) + J_z\sigma^z(i)\sigma^z(i+1)) \\
&= \sum_i J^+(\sigma^+(i)\sigma^-(i+1) + \sigma^-(i)\sigma^+(i+1)) + \\
&\quad \sum_i J^-(\sigma^+(i)\sigma^+(i+1) - \sigma^-(i)\sigma^-(i+1) + \sum_i J_z\sigma^z(i)\sigma^z(i+1)
\end{aligned}
$$

Now use Jordan Wigner

$c(i) = \prod_{j=1}^{i-1} \sigma^z(j)\sigma^-(i)$

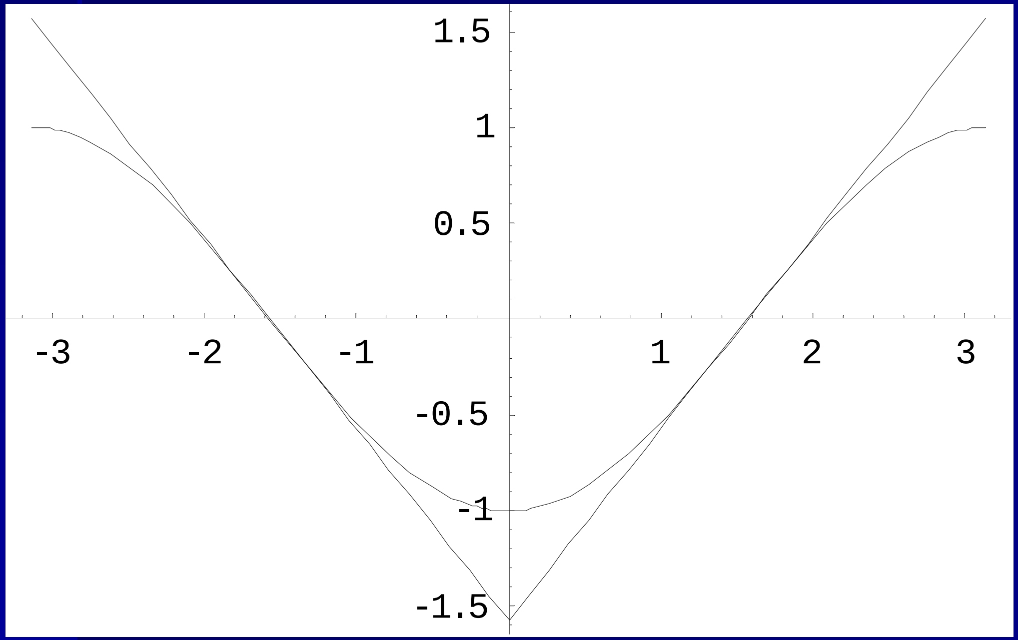
$n(i) = 1/2(\sigma^z(i) - 1)$

$H_{xyz} = \sum (c^\dagger(i)c(i+1) + c^\dagger(i+1)c(i)) + m_0 \sum (c^\dagger(i)c^\dagger(i+1) + c(i+1)c(i))) + \Delta \sum_i n(i)n(i+1)$

$m_0 = (J_x - J_y)$ and $\Delta = (J_x + J_y)$.

This is an interacting fermionic field theory (on a lattice) with a mass term, and reduces to Massive Thirring model in continuum limit.

$$H = \int dx \{ -i[\psi_1^\dagger \partial_x \psi_1 - \psi_2^\dagger \partial_x \psi_2] + m_0(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + 2g_0 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \}$$



It is easier and more general to work on lattice since we can take continuum limits easily!

Mark Kac's dictum:

Be wise ,

Discretize!

- Delta function bose/fermi gases (non relativistic)
- Heisenberg connection
- Bethe's Ansatz
- Calogero Sutherland system

$$H = \int dx\{[-\psi^\dagger(x)\partial_x^2\psi(x) + 2g_0(\psi^\dagger(x)\psi(x))^2\}$$

Bose gas with delta function (Lieb) or fermions with multicomponent (Yang Sutherland)

Can be viewed as a delta function interaction problem

$$H = -\sum_i \partial^2_{x(j)} + 2g\sum_{i,j}\delta(x(i) - x(j))$$

No need to show integrability (at least for bose case) since it can be btained by ANOTHER continuum limit of Heisenberg model. For fermions do need: Yang's consistency equations for 2 body scattering matrices:

SSS= SSS (will see in a minute with Baxter)

Bethe's Ansatz:

1933 !! For Heisenberg model (early bird special)

$$\psi(x_1, x_2, .., x_N) = \sum_P A_P \exp i(k_{P1}x_1 + k_{P2}x_2 + \; + k_{P_N}x_N)$$

Wave function parametrized by just N complex numbers k_1,..k_N and with a suitable amplitude A_P for each permutation.
Consistency check needed since hugely overdetermined equation set for these N parameters
Hints of integrability: We have N conserved numbers k's

Check for consistency by plugging in to Schroedinger equation: combinatorial problem.

S matrix for 2 body collisions is some ratio of A's

Matrix problem in case of Fermions since we need to impose statistics, where A_P needs to be generalized to different orderings of particles.
"One of the triumphs of Theoretical physics" (Yang)

Calogero Sutherland model and the Haldane Shastry- inverse square exchange model. (1970- 1989)

$$H_{CS} = -\sum \partial^2_{x(j)} + 2\sum_{i<j} \frac{\lambda(\lambda-1)}{\sin^2(x(i)-x(j))}$$

$$H_{HS} = \sum_{i<j} \frac{\vec{\sigma}(i).\vec{\sigma}(j)}{\sin^2(x(i)-x(j))}$$

Here a VERY detailed solution is available for 2 point functions unlike the Bethe case. Connections galore: Random matrix theory, Yangian symmetries, fractional statistics…….Once again, the discrete model solution gives the continuum model in the case $\lambda=2$ (but not in general). H_{CS} has as its ground state the various circular ensemble probability amplitudes!! So the CSE (symplectic) case has an extra SU(2) invariance that is mysterious.

So how about integrability?

Unlike classical mechanics, Lax eqn does not buy us much.

E.g. Calogero Sutherland case: Classical case J Moser

$$L_{i,j} = \delta_{i,j} p_i + \lambda(1-\delta_{i,j})(\cot(x(i) - \cot(x(j))$$
$$A_{i,j} = \delta_{i,j} \sum_k (\cot(x(i) - \cot(x(k)))^2 - (1-\delta_{i,j})(\cot(x(i) - \cot(x(j)))^2$$
$$[L_{i,j}, H] = [L, A]_{i,j}$$

But this does not buy integrability as such.

Extra ingredient needed:(Shastry-Sutherland 1992, Hikami 92)

$\Lambda_{i,j} = 1$ for all i, j. Clearly $A\Lambda = \Lambda A = 0$ if A has stochastic matrix structure as shown above. If so then $I_n \equiv Tr(L^n \Lambda)$ is conserved

Proof:using Quantum Lax + cyclicity of trace + stochastic property: $[I_n, H] = Tr\{L..[L,H]..L\Lambda\} = Tr\{L^n A\Lambda - Tr A L^n\}\Lambda = 0$

Or supersymmetric formulation of same result.

For discrete models: R J Baxter: Connection with statistical mechanics- Onsager's 2d Ising model generalization.

Object of interest is 1-d quantum operator " transfer matrix"

$$T(u) = Tr_g[L_{N,g}(u)L_{N-1,g}(u).L_{1,g}(u)]$$

Where $L_{n,g}(u)$ are 2 body scattering matrices dependent on a spectral parameter u, and g is an auxiliary space variable (e.g. 2 dimensions) so this is an ordered product, much like a time evolution operator in QM. As $u \to 0$ one finds from general grounds $T(u) = T(0)\{1 + uH + u^2 H_2 + ..\}$ or exponentiating

$$T(u) = T(0)\exp(\{1 + uH + u^2 J_2 + u^2 J_3..)$$

. Baxter argued brilliantly that commuting transfer matrices suffice to give integrability and also explicit solutions (inspired by Onsager)

$$[T(u), T(v)] = 0$$

Follows from local relation (Yang Baxter Eqn)

$$L_{n,g1}(u)L_{n,g2}(v)R_{g1,g2}(u-v) = R_{g1,g2}(u-v)L_{n,g1}(v)L_{n,g2}(u)$$

This is the most important equation in integrable systems theory!

For u$\sim$ v, one can expand and get Lax equation etc..

This clearly shows that one has **``infinite''** conservation laws!

One can find various representations of this and related algebra: Drinfeld (Fields medal). Clearly higher conservation laws are just J2 J3 etc (log derivatives of the transfer matrix).

Usually R is a function of the difference of the spectral parameters, or uniformization is possible. However, this fails for the Hubbard model.

Hubbard model: Bethe's ansatz Lieb Wu (1970)

NON PERTURBATIVE MOTT INSULATING SPIN LIQUID GROUND STATE

Integrability, conservation laws and R matrix Shastry (1986). Long delay due to a ``red herring'', no uniformization possible for Hubbard model.

$$H = -t \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^{\dagger}_{j\,s} c_{j+1\,s} + c^{\dagger}_{j+1\,s} c_{j\,s}) + U \sum_{j=1}^{N} \left(n_{j\uparrow} - \frac{1}{2}\right)\left(n_{j\downarrow} - \frac{1}{2}\right) \qquad (1)$$

Finite in all respects:

Energy level variation with parameter

H=H(U) so all levels (discrete) vary with U.

Wigner von Neumann non crossing rule:

For a real symmetric (hermitean) matrix, we need to tune two (three) parameters to get a level crossing. Hence expect non crossing or level repulsion when two levels approach each other:

$$\Delta = \sqrt{(E_1 - E_2)^2 + |V_{1,2}|^2}$$

Two (three) parameters to be fine tuned are $E_1 - E_2$, and $V_{1,2}$ (and $V^*_{1,2}$)

Quantum Chaos Dogma: 1980's P(s) is distribution of level spacings

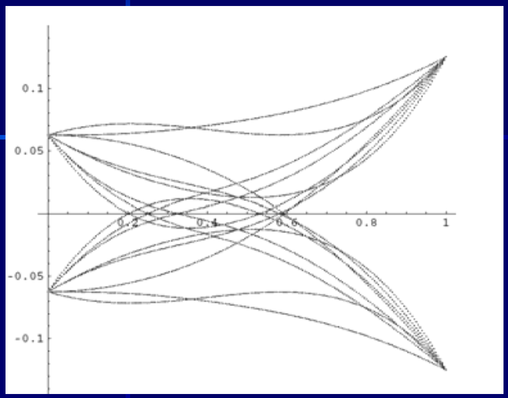
Level separations cluster at zero for "regular" i.e. integrable spectra, but repel for "irregular " i.e. on integrable spectra.

Percival, Michael Berry, Oriol Bohigas,…

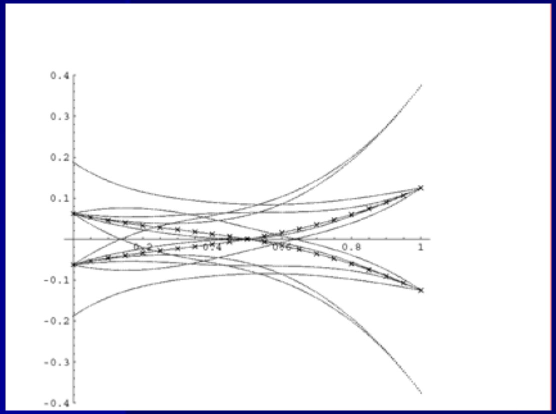
Poisson distribution of P(s) for integrable models

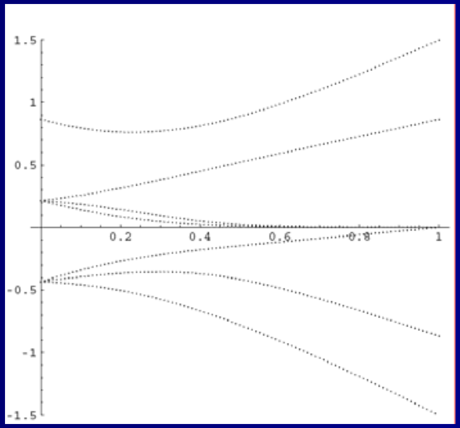
versus level repulsion P(s)= s exp(-s^2) (Wigner surmise)

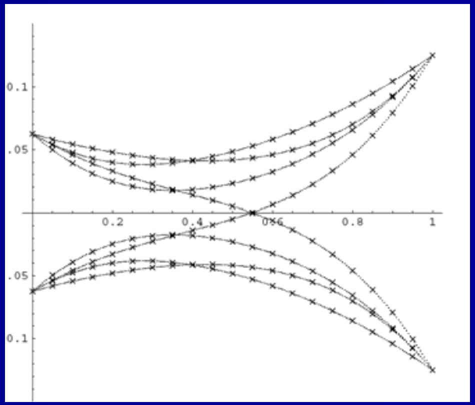
Numerical study of Hubbard model Heilmann Lieb/ Yuzbashyan Altshuler Shastry



U--independent symmetries fall into three major categories -- the symmetry of the polygon, the spin symmetry, and the particle-hole symmetry. In each figure we extract ALL known symmetries So we expect NO level crossings. And what we find instead is:







Questions in finite dimensions:

- **Is it meaningful to talk about integrability in finite dimensional spaces? Projection operators**
- **Given a finite dimensional matrix, can we recognize it as being integrable? Example below:**
- **Can we talk about integrability even when the notion of “degree of freedom” is ambiguous?**

Given arbitrary H, compute eigenbasis $H|j>= E_j|j>$ and $P_j = |j>< j|$ so that $[H, P_j] = 0 = [P_i, P_j]$.

So the skeptic argued, what is the big deal about integrabiity, everybody is integrable!!

Out[32]=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} + U \begin{pmatrix} \frac{1}{6} & \frac{1}{3\sqrt{2}} & \frac{1}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix}$$

Conceptual Answer: parameter dependence and Wigner von Neumann violation is key definition of integrability. Poisson statistics also implies a forgiveness of level crossings, (but requires ensembles to define).

Recent preprint: SS cond-mat/0501502 "A class of parameter dependent commuting matrices"

Think dynamical symmetries: $\alpha(x) = a + xA, \;\; \beta(x) = b + xB$

Idea: α is a ``Hamiltonian'' that possesses a dynamical symmetry β

Here a is a diagonal matrix with diagonal entries $\{a_1, a_2, ..., a_d\}$ and A is a real symmetric matrix. In the matrix α we have $d(d-1)/2$ off diagonal variables $A_{i,j}$, d variables a_j and a further d variables $A_j \equiv A_{j,j}$, in addition to the real variable x . Likewise with b and B. we have $\{b_1, b_2, ..., b_d\}$ and $\{B_1, ...B_d\}$ as well as $B_{i,j}$ with $i < j$.

Commutation for all x gives us two equations

$$\begin{aligned} [a, B] &= [b, A] \\ [A, B] &= 0. \end{aligned}$$

$$\Rightarrow S_{i,j} = \frac{A_{i,j}}{a_i - a_j} = \frac{B_{i,j}}{b_i - b_j}$$

2 x d(d-1)2 eqns

$$\Rightarrow Y_{i,j}[\alpha] = Y_{i,j}[\beta]$$

$$Y_{i,j}[\alpha] \equiv \frac{A_i - A_j}{a_i - a_j} - \frac{1}{S_{i,j}(a_i - a_j)} \sum_{l \neq i,j} S_{i,l} S_{l,j} (a_l - a_j)$$

$\rightarrow Y_{i,j}[\alpha] = Y_{i,j}[\beta]$

Symmetric in α and β matrices. Now solve for β in terms of α

$$\xi_{i,j} \equiv B_i - B_j = p_{i,j}(b_i - b_j) + \frac{1}{a_i - a_j} \sum_{l \neq i,j} \frac{S_{i,l} S_{l,j}}{S_{i,j}} (b \wedge a)_{i,j,l}$$

$$(b \wedge a)_{i,j,l} \equiv [(b_i a_j - b_j a_i) + (b_j a_l - b_l a_j) + (b_l a_i - b_i a_l)]$$

IF α is assumed known then above d(d-1)/2 linear eqns are for 2 d variables $\{b_1, b_2.., b_d\}, \{B_1, B_2, .., B_d\}$. OVERDETERMINED.
Consistency requires triangle law

$$\xi_{i,j} + \xi_{j,k} + \xi_{k,i} = 0$$

$$LHS = \mu(i; j, k) b_i + \mu(j; k, i) b_j + \mu(k; i, j) b_k + \sum_{l \neq i,j,k} \nu(l; i, j, k) b_l$$

Type I solutions: (Think Bethe's Ansatz). Coefficient of each b_r identicall zero.

Type II rest: (mostly trivial such as α=β)

The task

To prove that Type I solutions exist!

Highly non trivial since

- For each distinct set of indices {i,j,k} each μ(i,j,k) must vanish (~d^3/6 eqns)
- For each distinct {i,j,k,l}, each ν(i,j,k,l) must vanish (~d^4/24 eqns)
- Number of variables: S_{i,j} (d(d-1)/2) in number and {a_j's} {A_j's} 2d in number.

And yet very tempting since α's are being determined without any knowledge of β's, other than the fact that they exist! This is reminiscent of definition of integrability. Principle of AUTONOMY

$$S_{i,l}S_{l,j}S_{j,k}S_{k,i} + S_{j,l}S_{l,k}S_{k,i}S_{i,j} + S_{k,l}S_{l,i}S_{i,j}S_{j,k} = 0$$

Autonomous eqn for S's for fixed indices. These correspond to three Hamilton walks on the tetrahedron (oriented). Become simpler for Symmetric Has compared to Hermitean!

Formidable task, and yet for case of real symmetric matrices it **all** works out:

What do the solutions look like, what are their properties and what do they mean?

I believe that these essentially give a clue to discovering all integrable models in finite dimensions. Fine tuning is still needed but this is a sizable step.

Final result: (2 d -3) free parameters for determining S's and (d+2) free parameters for determining a , A. Rest are to be found from given algebraic equations. Given α, we have the choice of fixing (d+1) parameters to fix $\beta(x)$ matrices which commute with $\alpha(x)$.

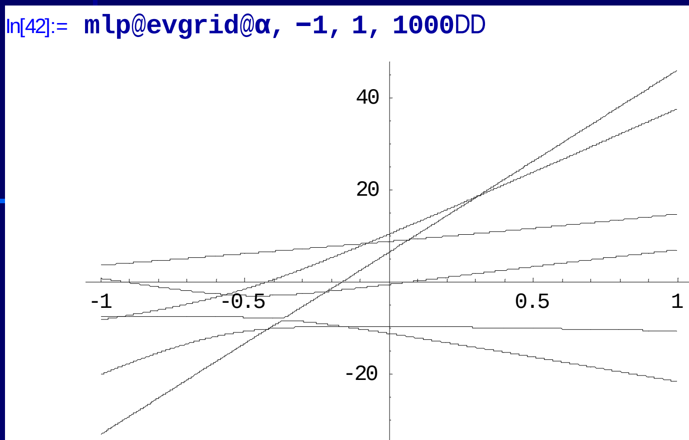
Properties:

If we take any Type I matrix $\alpha(x)$, and constructed commuting $\beta(x)$, then these invariably violate Wigner von Neumann non crossing rules.

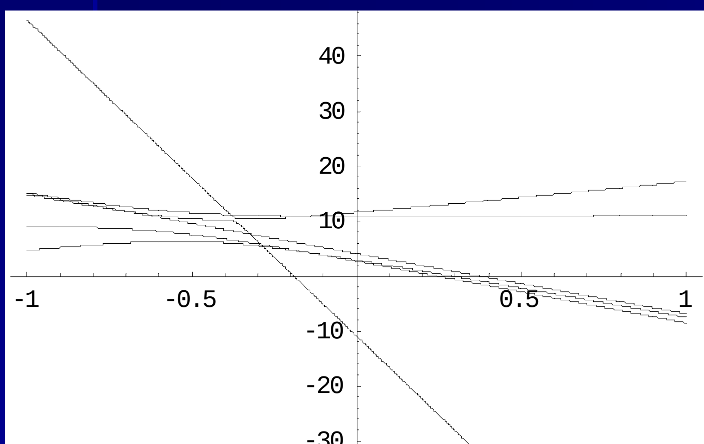
These define classes of matrices for a fixed S matrix, and partners are automatically of Type I i.e. What is α and what is β are freely interchangable, as in integrable models.

Numerical results from notebook available at physics.ucsc.edu/~sriram/demo_shastry.nb

α

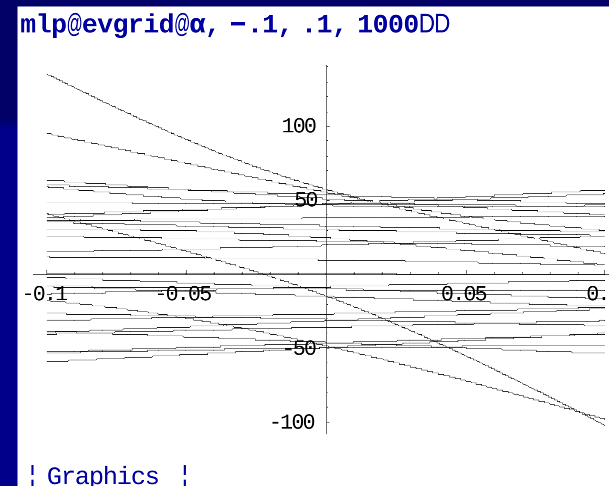


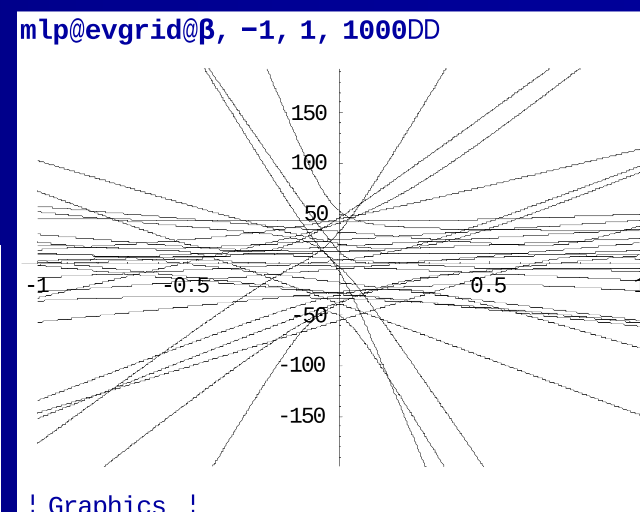
β



d=30
->

α





β

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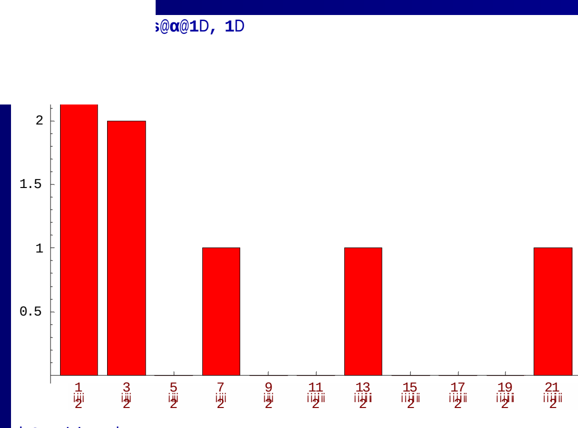
d=6

P(s)



Summary and open questions:

•We can display matrices in any dimension that depend on a parameter and commute within a well defined class

•This seems to produce violations of repulsion of levels due to Wigner and von Neumann

•Algebraic proof of the connection between commutation and level crossings is possible in d=3 (exactly) but not yet available in higher dimensions.

•Does this class contain all integrable systems? Work needs to be done on this.