Neutrino Decoupling and Big Bang Nucleosynthesis, Photon Decoupling, and WIMP Annihilation

![Diagram](image.png)

*Fig. 3.1. The thermal history of the standard model. The densities of protons, electrons, photons, and neutrinos are shown at various stages of cosmological evolution [after Harrison (1973)]*

The Planck Mass is

The Planck Length

\[ l_{Pl} = \sqrt{\frac{hG}{2\pi c^3}} = 1.6 \times 10^{-33} \text{ cm} \]

is the smallest possible length. Here \( h \) is Planck’s constant

\[ h = 6.626068 \times 10^{-34} \text{ m}^2 \text{ kg/ s} \]

The Planck Mass is

\[ m_{Pl} = \sqrt{\frac{hc}{2\pi G}} = 2.2 \times 10^{-5} \text{ g} \]

The Compton (i.e. quantum) wavelength

\[ l_C = \frac{h}{2\pi mc} \]
equals the Schwarzschild radius

\[ l_S \approx \frac{Gm}{c^2} \]

when \( m = m_{Pl} = 1.2 \times 10^{19} \text{ GeV/ c}^2 \)

From *The View from the Center of the Universe* © 2006
“Natural Units” = High Energy Physics Units \( \hbar = c = k_B = 1 \)

These are especially appropriate for the hot early universe. There is one fundamental dimension, which we can take to be mass or energy = \( mc^2 = m \).

\[
[\text{Energy}] = [\text{Mass}] = [\text{Temperature}] = [\text{Length}]^{-1} = [\text{Time}]^{-1}
\]

\[1 \text{ GeV} = 1.78 \times 10^{-24} \text{ g} = 1.16 \times 10^{13} \text{ K} = 1.97 \times 10^{-14} \text{ cm}^{-1} = 6.58 \times 10^{-25} \text{ s}^{-1} = 1.602 \times 10^{-3} \text{ erg} \]

\[1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} = 1.16 \times 10^{4} \text{ K} \quad (k_B = 1.38 \times 10^{-16} \text{ erg K}^{-1}) \]

\[1 \\text{ pc} = 3.26 \text{ lyr} = 3.09 \times 10^{18} \text{ cm} \]

To add gravity to this scheme, we usually express it in terms of the Planck mass \( M_{\text{Pl}} = (\hbar c/G)^{1/2} = 1.2 \times 10^{19} \text{ GeV}/c^2 \)

For more conversion factors, see Appendix A of Kolb & Turner, *The Early Universe*
Big Bang Nucleosynthesis

BBN was conceived by Gamow in 1946 as an explanation for the formation of all the elements, but the absence of any stable nuclei with A=5,8 makes it impossible for BBN to proceed past Li. The formation of carbon and heavier elements occurs instead through the triple-\(\alpha\) process in the centers of red giants (Burbidge\(^2\), Fowler, & Hoyle 57). At the BBN baryon density of \(2 \times 10^{-29} \Omega_b h^2 (T/T_0)^3 \text{ g cm}^{-3} \approx 2 \times 10^{-5} \text{ g cm}^{-3}\), the probability of the triple-\(\alpha\) process is negligible even though \(T \approx 10^9\text{K}\).

Thermal equilibrium between n and p is maintained by weak interactions, which keeps \(n/p = \exp(-Q/T)\) (where \(Q = m_n - m_p = 1.293\text{ MeV}\)) until about \(t \approx 1\text{ s}\). But because the neutrino mean free time \(t_{\nu}^{-1} \approx \sigma_{\nu} n_e \approx (G_F T)^2 T^3\) is increasing as \(t_{\nu} \propto T^{-5}\) (here the Fermi constant \(G_F \approx 10^{-5}\text{ GeV}^{-2}\)), while the horizon size is increasing only as \(t_H \approx (G \rho)^{-1/2} \approx M_{\text{Pl}} T^{-2}\), these interactions freeze out when T drops below about 0.8 MeV. This leaves \(n/(p+n) \approx 0.14\). The neutrons then decay with a mean lifetime \(887 \pm 2\text{ s}\) until they are mostly fused into D and then \(^4\text{He}\). The higher the baryon density, the higher the final abundance of \(^4\text{He}\) and the lower the abundance of D that survives this fusion process. Since D/H is so sensitive to baryon density, David Schramm called deuterium the “baryometer.” He and his colleagues also pointed out that since the horizon size increases more slowly with \(T^{-1}\) the larger the number of light neutrino species \(N_\nu\) contributing to the energy density \(\rho\), BBN predicted that \(N_\nu \approx 3\) before \(N_\nu\) was measured at accelerators by measuring the width of the \(Z^0\) (Cyburt et al. 2005: \(2.67 < N < 3.85\)).
THE NUMBER OF LIGHT NEUTRINO TYPES FROM COLLIDER EXPERIMENTS

Revised March 2008 by D. Karlen (University of Victoria and TRIUMF).

The most precise measurements of the number of light neutrino types, $N_\nu$, come from studies of $Z$ production in $e^+e^-$ collisions. The invisible partial width, $\Gamma_{\text{inv}}$, is determined by subtracting the measured visible partial widths, corresponding to $Z$ decays into quarks and charged leptons, from the total $Z$ width. The invisible width is assumed to be due to $N_\nu$ light neutrino species each contributing the neutrino partial width $\Gamma_\nu$ as given by the Standard Model. In order to reduce the model dependence, the Standard Model value for the ratio of the neutrino to charged leptonic partial widths, $(\Gamma_\nu/\Gamma_\ell)_{\text{SM}} = 1.991\pm0.001$, is used instead of $(\Gamma_\nu)_{\text{SM}}$ to determine the number of light neutrino types:

$$N_\nu = \frac{\Gamma_{\text{inv}}}{\Gamma_\ell} \left( \frac{\Gamma_\ell}{\Gamma_\nu} \right)_{\text{SM}}.$$

The combined result from the four LEP experiments is $N_\nu = 2.984\pm0.008$ [1].

In the past, when only small samples of $Z$ decays had been recorded by the LEP experiments and by the Mark II at SLC, the uncertainty in $N_\nu$ was reduced by using Standard Model fits to the measured hadronic cross sections at several center-of-mass energies near the $Z$ resonance. Since this method is much more dependent on the Standard Model, the approach described above is favored.

Comparison of $N_\nu$ constraints using various data set combinations. “All” refers to WMAP3 + other CMB + Ly$\alpha$ + galaxy power spectrum (SDSS main sample + 2dF) + SDSS baryon acoustic oscillation (BAO) + Supernovae Ia (SN). See Ref. [4] for details. SDSS (main) and Ly$\alpha$ favor $N_\nu > 3$. 
Neutrinos in the Early Universe

As we discussed, neutrino decoupling occurs at $T \sim 1$ MeV. After decoupling, the neutrino phase space distribution is

$$f_{\nu} = \left[1 + \exp\left(\frac{p_{\nu}c}{T_{\nu}}\right)\right]^{-1} \quad \text{(note: } \neq \left[1 + \exp\left(\frac{E_{\nu}}{T_{\nu}}\right)\right]^{-1} \text{ for NR neutrinos)}$$

After $e^+e^-$ annihilation, $T_{\nu} = (4/11)^{1/3}T_\gamma = 1.9K$. Proof:

Number densities of primordial particles

$$n_\gamma(T) = 2 \zeta(3) \pi^2 T^3 = 400 \text{ cm}^{-3} (T/2.7K)^3, \quad n_\nu(T) = \left(\frac{3}{4}\right) n_\gamma(T) \text{ including antineutrinos}$$

Conservation of entropy $s_i$ of interacting particles per comoving volume

$$s_i = g_i(T) N_\gamma(T) = \text{constant}, \text{ where } N_\gamma = n_\gamma V; \text{ we only include neutrinos for } T > 1 \text{ MeV.}$$

Thus for $T > 1$ MeV, $g_i = 2 + 4(7/8) + 6(7/8) = 43/4$ for $\gamma$, $e^+e^-$, and the three $\nu$ species, while for $T < 1$ MeV, $g_i = 2 + 4(7/8) = 11/2$. At $e^+e^-$ annihilation, below about $T = 0.5$ Mev, $g_i$ drops to 2, so that

$$2N_{\gamma 0} = g_i(T < 1 \text{ MeV}) N_\gamma(T < 1 \text{ MeV}) = (11/2) N_\gamma(T < 1 \text{ MeV}) = (11/2)(4/3) N_\nu(T < 1 \text{ MeV}).$$

Thus $n_{\nu 0} = \left(\frac{3}{4}\right)(4/11) n_{\gamma 0} = 109 \text{ cm}^{-3} (T/2.7K)^3$, or

$$T_\nu = (4/11)^{1/3} T = 0.714 T$$
Statistical Thermodynamics

\[ n_i = \frac{g_i}{2m_i} \left( \frac{kT_i}{\hbar c} \right)^3 I_i''(\pm), \quad \rho_i = \frac{2i}{2\pi^2 c^2} \left( \frac{kT_i}{\hbar c} \right)^3 I_i''(\pm), \quad \text{where} \]

\[ I_i^{mn} = \int_0^{2\pi} \int_0^{\infty} (x^2 - \theta_i^2)^{n/2} (e^{x^2} + 1)^{-1} dx, \quad \theta_i = \frac{kT_i}{m_i c^2}, \quad g_i = \text{# spin states} \]

+ Fermi-Dirac, - Bose-Einstein

\[ \theta_i \gg 1 \quad (ER): \quad I_i''(\pm) = \frac{3}{2} S(3) = 1.803, \quad I_i^{21}(\pm) = \frac{7\pi^4}{120} \]

\[ \theta_i \ll 1 \quad (NR): \quad n_i = \frac{\rho_i}{m_i} = \frac{g_i}{(2\pi)^{3/2}} \left( \frac{kT_i}{\hbar c} \right)^3 \theta_i^{-3/2} e^{-\theta_i} \quad (\mu \text{ in } \nu's) \]

\[ \text{Note: } S(3) = 1.2020569... = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{k^3} \right)^{-1} \]
Boltzmann Equation

In the absence of interactions (rhs=0) \( n_1 \) falls as \( a^{-3} \)

We will typically be interested in \( T >> E_\mu \) (where \( \mu \) is the chemical potential). In this limit, the exponential in the Fermi-Dirac or Bose-Einstein distributions is much larger than the \( \pm 1 \) in the denominator, so that

\[
f(E) \to e^{\mu/T}e^{-E/T}
\]

and the last line of the Boltzmann equation above simplifies to

\[
f_3f_4[1 \pm f_1][1 \pm f_2] - f_1f_2[1 \pm f_3][1 \pm f_4] \to e^{-(E_1+E_2)/T}\left\{ e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right\}.
\]

The number densities are given by

<table>
<thead>
<tr>
<th>Neutron-Proton Ratio</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutron-Proton Ratio</td>
<td>( n )</td>
<td>( \nu_e ) or ( e^+ )</td>
<td>( p )</td>
<td>( e^- ) or ( \bar{\nu}_e )</td>
</tr>
<tr>
<td>Recombination</td>
<td>( e )</td>
<td>( p )</td>
<td>( H )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>Dark Matter Production</td>
<td>( X )</td>
<td>( X )</td>
<td>( l )</td>
<td>( l )</td>
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</tbody>
</table>

For our applications, i's are
The equilibrium number densities are given by

\[ n_i^{(0)} \equiv g_i \int \frac{d^3 p}{(2\pi)^3} e^{-E_i/T} = \begin{cases} g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T} & \text{if } m_i \gg T \\ g_i \frac{T^3}{\pi^2} & \text{if } m_i \ll T \end{cases} \]  

(3.6)

With this definition, \( e^{\mu_i/T} \) can be rewritten as \( n_i/n_i^{(0)} \), so the last line of Eq. (3.1) is equal to

\[ e^{-(E_1 + E_2)/T} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \]  

(3.7)

With these approximations the Boltzmann equation now simplifies enormously. Define the thermally averaged cross section as

\[ \langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4} e^{-(E_1 + E_2)/T} \]

\[ \times (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4) \delta(E_1 + E_2 - E_3 - E_4) |\mathcal{M}|^2. \]  

(3.8)

Then, the Boltzmann equation becomes

\[ a^{-3} \frac{d}{dt} \left( n_1 a^3 \right) = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}. \]  

(3.9)

If the reaction rate \( n_2 \langle \sigma v \rangle \) is much smaller than the expansion rate (~ H), then the \{\} on the rhs must vanish. This is called \textit{chemical equilibrium} in the context of the early universe, \textit{nuclear statistical equilibrium} (NSE) in the context of Big Bang nucleosynthesis, and the \textit{Saha equation} when discussing recombination of electrons and protons to form neutral hydrogen.