Homework Set 2
DUE: Thursday May 8

1. **Power Spectrum of Perturbations.** Suppose that the power spectrum of density perturbations $\delta \equiv (\delta \rho / \rho)$, with index $n$, is $P_k \equiv |\delta_k|^2 \propto k^n$.

a) Derive this expression for the rms density fluctuations on a comoving scale $\lambda$:

$$\left( \frac{\delta \rho}{\rho} \right)_\lambda \sim k^{3/2} |\delta_k| \propto \lambda^p$$

and find the index $p$ in terms of $n$.

b) Find the index $m$ for the rms mass fluctuations on a comoving scale $\lambda$:

$$(\delta M/M)_\lambda \propto M^m.$$ 

What is the criterion on $n$ such that structures grow in a top-down manner (large masses collapse first)? In a bottom-up manner? If the primordial spectrum has $n = 1$ and the CDM transfer function goes as $T(k) \sim k^{-2}$ due to the horizon entry effect, what $m$ does this correspond to and what is the physical interpretation of the mass fluctuation spectrum?

c) The dynamics of the perturbations are driven by the fluctuations in the gravitational potential $\phi$. Find the index $f$ for the rms potential fluctuations

$$\delta \phi \lambda \propto \lambda^f.$$ 

What is the physical significance of the scale invariant power spectrum index $n = 1$? Also explain the difference between a process being scale free and being scale invariant. Which, if either, is Newtonian gravity?

2. **Growth of Density Perturbations.** For matter density perturbations with wavelengths much greater than the Jeans length, the time evolution is given by

$$\ddot{\delta} + 2H\dot{\delta} - (3/2)\Omega_m(t) H^2 \delta = 0.$$ 

a) Rewrite this equation with the dependent variable being the scale factor $a$. Write any derivatives of $a$ or $H$ in terms of $H$ and the deceleration parameter $q$. b) Consider the case of a flat matter universe where, on the scales considered, only a constant fraction $\Omega_{cl}$ clumps to form structure. (Possible realizations are a cold + hot dark matter universe or a dark + baryonic matter universe.) In this case the $\Omega_m$ in the source term of the evolution equation is replaced by $\Omega_{cl}$. Solve the equation to find the behavior of the growing mode: $\delta \propto a^m$. Interpret. Check the limits $\Omega_{cl} = 0$ and 1. c) Consider the case of an open
universe at a time dominated by the curvature. Write the evolution equation, substituting in for \( q \) and \( \Omega_m(t) \), keeping only the leading order for the coefficient of each term as \( a \) gets large. What happens to the source term for the growth as the universe expands? Try a solution \( \delta \propto a^m \). Find the dominant mode and give the physical interpretation (include explanation of the roles of both the drag and source terms).

3. Spherical Collapse. The evolution of a spherically symmetric, overdense perturbation in an \( \Omega = 1 \) universe can be solved analytically up to the point of singular collapse. As a consequence of Birkhoff’s theorem (in a spherically symmetric universe, only the interior mass matters), the perturbation follows the equations of a \( k = +1 \) Friedmann universe, for which we have a parametric solution. a) Perturbation overdensity The solutions – unperturbed for the background universe (barred quantities) and perturbed for the overdense region – for the evolution of the size of a sphere and the time are given by

\[
\bar{r} = r_0 \left( \frac{a}{a_0} \right) = r_0 \left( \frac{\eta}{\eta_0} \right)^2 \\
\bar{t} = t_0 \left( \frac{a}{a_0} \right)^{3/2} = t_0 \left( \frac{\eta}{\eta_0} \right)^3 \\
r = A (1 - \cos \theta), \\
t = B (\theta - \sin \theta),
\]

where \( \eta \) is the conformal time and \( \theta \) is the development angle.

At early times the density perturbation must be small (\( \rho \to \bar{\rho} \)) so the Friedmann equations for the universe and the perturbation region look the same. Enforce this by matching \( x, \dot{x}, \ddot{x} \) for \( x = r, \bar{r} \) with the respective time variables in order to find \( r(\theta) \) and \( t(\theta) \), i.e. \( A \) and \( B \). Hint: Remember the definition of the conformal time parameters \( \eta \) and \( \theta \).

The age of the universe is unique so \( t \) and \( \bar{t} \) must be equal. Use this to derive \( \eta(\theta) \).

Use mass conservation to express the overdensity \( \rho/\bar{\rho} \) first in terms of \( r/\bar{r} \) and then as a function of \( \theta \).

Verify that turnaround occurs at \( \theta = \pi \) and \( \rho/\bar{\rho} = 5.55 \) and that virialization occurs at \( \rho/\bar{\rho} = 178 \). Use that the radius is half the turnaround radius (implying \( V = -2K \)), but use the time corresponding to \( \theta = 2\pi \). (Although \( V = -2K \) at \( \theta = \pi/2 \), virialization requires \( \langle V \rangle = -2\langle K \rangle \), which obtains roughly at \( \theta = 2\pi \)).

b) Linear regime Show that the density contrast

\[
\frac{\delta \rho}{\rho} = \frac{\rho - \bar{\rho}}{\bar{\rho}} \propto a
\]

when \( \theta \ll 1 \). Show that the dimensionless velocity perturbation for \( \theta \ll 1 \) is

\[
\delta_v \equiv \frac{v - Hr}{Hr} = -\frac{1}{3} \left( \frac{\delta \rho}{\rho} \right),
\]

where \( v = dr/dt \) is the perturbation’s expansion velocity and \( H \) is the Hubble parameter of the background universe.

c) Astrophysical application Suppose that we observe a galaxy with rotation speed \( \sigma \) at radius \( R \). If we attribute this rotation speed to the mass of a (spherical) dark halo and assume the spherical collapse model in an \( \Omega = 1 \) universe gives an accurate description of the formation of this halo, what is the expression for the redshift of virialization \( z_v \)? What is the value of \( z_v \) if \( \sigma = 180 \) km s\(^{-1} \), \( R = 30 \) kpc and \( H_0 = 60 \) km s\(^{-1} \) Mpc\(^{-1} \)?